



# ***On canonical equation for water waves***

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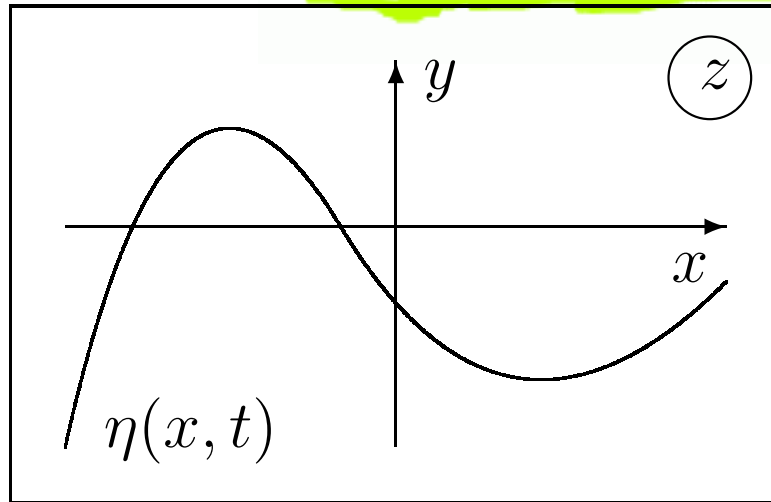
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# Equations



potential irrotational flow

$$\Delta\phi(x, y, t) = 0$$

Boundary conditions:

$$\left[ \begin{array}{l} \frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = \frac{P}{\rho}, \\ \frac{\partial\eta}{\partial t} + \eta_x\phi_x = \phi_y \end{array} \right] \text{ at } y = \eta(x, t).$$

$$\frac{\partial\phi}{\partial y} = 0, y \rightarrow -\infty,$$

$$\frac{\partial\phi}{\partial x} = 0, |x| \rightarrow \infty.$$

# Hamiltonian

Hamiltonian  $H$  is the total energy of the fluid  $H = T + U$

$$\begin{aligned} T &= \frac{1}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\eta} (\nabla \Phi)^2 dy, & \frac{\partial \eta}{\partial t} &= \frac{\delta H}{\delta \Psi}, \\ U &= \frac{g}{2} \int \eta^2 dx. & \frac{\partial \Psi}{\partial t} &= -\frac{\delta H}{\delta \eta}, \\ & & \Psi(x, t) &= \Phi(x, y, t)|_{y=\eta} \end{aligned}$$

$$H = \frac{1}{2} \int_{-\infty}^{\infty} \Psi \hat{G}(\eta) \Psi dx + \frac{g}{2} \int_{-\infty}^{\infty} \eta^2 dx$$

# Classical variables $\Psi, \eta$

Normal complex variable  $a_k$ :

$$\eta_k = \sqrt{\frac{\omega_k}{2g}} (a_k + a_{-k}^*) \quad \psi_k = -i \sqrt{\frac{g}{2\omega_k}} (a_k - a_{-k}^*) \quad \omega_k = \sqrt{gk}$$

$$\begin{aligned} \mathcal{H} &= \int \omega_k |a_k|^2 + \int V_{k_1 k_2}^k \{a_k^* a_{k_1} a_{k_2} + a_k a_{k_1}^* a_{k_2}^*\} \delta_{k-k_1-k_2} dk dk_1 dk_2 \\ &+ \frac{1}{3} \int U_{k k_1 k_2} \{a_k a_{k_1} a_{k_2} + a_k^* a_{k_1}^* a_{k_2}^*\} \delta_{k+k_1+k_2} dk dk_1 dk_2 + \\ &+ \frac{1}{4} \int W_{k_2 k_3}^{k k_1} a_k^* a_{k_1}^* a_{k_2} a_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3 \end{aligned}$$

## Normal variables $a_k$

$a_k$  satisfies the equation

$$\frac{\partial a_k}{\partial t} + i \frac{\delta H}{\delta a_k^*} = 0,$$

Three wave resonances are absent

$$\begin{aligned} k &= k_1 + k_2, \\ \omega_k &= \omega_{k_1} + \omega_{k_2}, \end{aligned}$$

**NO!**

Cubic nonresonant terms can be excluded by canonical transformation:

$$a_k \rightarrow b_k.$$

# Transformation $a_k \rightarrow b_k$

$$a_k = b_k + \int \Gamma_{k_1 k_2}^k b_{k_1} b_{k_2} \delta_{k-k_1-k_2} - 2 \int \Gamma_{k k_1}^{k_2} b_{k_1}^* b_{k_2} \delta_{k+k_1-k_2} +$$

$$+ \int \Gamma_{k k_1 k_2} b_{k_1}^* b_{k_2}^* \delta_{k+k_1+k_2} + \int B_{k_2 k_3}^{k k_1} b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} +$$

$$B_{k_2 k_3}^{k k_1} = \Gamma_{k_2 k_1 - k_2}^{k_1} \Gamma_{k k_3 - k}^{k_3} + \Gamma_{k_3 k_1 - k_3}^{k_1} \Gamma_{k k_2 - k}^{k_2} -$$

$$- \Gamma_{k_2 k - k_2}^k \Gamma_{k_1 k_3 - k_1}^{k_3} - \Gamma_{k_3 k_1 - k_3}^{k_1} \Gamma_{k_1 k_2 - k_1}^{k_2} -$$

$$- \Gamma_{k k_1}^{k+k_1} \Gamma_{k_2 k_3}^{k_2+k_3} + \Gamma_{-k-k_1 k k_1} \Gamma_{-k_2-k_3 k_2 k_3} + \tilde{B}_{k_2 k_3}^{k k_1}.$$

$\tilde{B}_{k_2 k_3}^{k k_1}$  is arbitrary function, satisfying symmetry conditions:

$$\tilde{B}_{k_2 k_3}^{k k_1} = \tilde{B}_{k_2 k_3}^{k_1 k} = \tilde{B}_{k_3 k_2}^{k k_1} = -(\tilde{B}_{k k_1}^{k_2 k_3})^*.$$

This transformation is canonical up to the order of  $|b_k|^4$ .

# Classical variables $b_k$

Here

$$\Gamma_{k_1 k_2}^k = -\frac{1}{2} \frac{V_{k_1 k_2}^k}{\omega_k - \omega_{k_1} - \omega_{k_2}} \quad \Gamma_{k k_1 k_2} = -\frac{1}{2} \frac{U_{k k_1 k_2}}{\omega_k + \omega_{k_1} + \omega_{k_2}}.$$

Simplest choice  $\tilde{B}_{k_2 k_3}^{k k_1} = 0$ .

$$\mathcal{H} = \int \omega_k b_k b_k^* dk + \frac{1}{4} \int T_{k k_1}^{k_2 k_3} b_k^* b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3 + \dots;$$

# Zakharov equation

$$i\dot{b} = \omega_k b_k + \frac{1}{2} \int T_{kk_1}^{k_2 k_3} b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3$$



# Zakharov equation

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$$\begin{aligned}
 T_{k_2 k_3}^{k k_1} = & W_{k_1 k}^{k_2 k_3} - \\
 & - V_{k_2 k - k_2}^k V_{k_1 k_3 - k_1}^{k_3} \left[ \frac{1}{\omega_{k_2} + \omega_{k - k_2} - \omega_k} + \frac{1}{\omega_{k_1} + \omega_{k_3 - k_1} - \omega_{k_3}} \right] - \\
 & - V_{k_2 k_1 - k_2}^{k_1} V_{k k_3 - k}^{k_3} \left[ \frac{1}{\omega_{k_2} + \omega_{k_1 - k_2} - \omega_{k_1}} + \frac{1}{\omega_k + \omega_{k_3 - k} - \omega_{k_3}} \right] - \\
 & - V_{k_3 k - k_3}^k V_{k_1 k_2 - k_1}^{k_2} \left[ \frac{1}{\omega_{k_3} + \omega_{k - k_3} - \omega_k} + \frac{1}{\omega_{k_1} + \omega_{k_2 - k_1} - \omega_{k_2}} \right] - \\
 & - V_{k_3 k_1 - k_3}^{k_1} V_{k k_2 - k}^{k_2} \left[ \frac{1}{\omega_{k_3} + \omega_{k_1 - k_3} - \omega_{k_1}} + \frac{1}{\omega_k + \omega_{k_2 - k} - \omega_{k_2}} \right] - \\
 & - V_{k k_1}^{k+k_1} V_{k_2 k_3}^{k_2+k_3} \left[ \frac{1}{\omega_{k+k_1} - \omega_k - \omega_{k_1}} + \frac{1}{\omega_{k_2+k_3} - \omega_{k_2} - \omega_{k_3}} \right] - \\
 & - U_{-k-k_1 k k_1} U_{-k_2-k_3 k_2 k_3} \left[ \frac{1}{\omega_{k+k_1} + \omega_k + \omega_{k_1}} + \frac{1}{\omega_{k_2+k_3} + \omega_{k_2} + \omega_{k_3}} \right]
 \end{aligned}$$

$T_{kk_1}^{k_2k_3}$  **vanishes**

On the resonant manifold

$$k + k_1 = k_2 + k_3,$$

$$\omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3},$$

$$k = a(1 + \zeta)^2,$$

$$k_1 = a(1 + \zeta)^2 \zeta^2,$$

$$k_2 = -a\zeta^2,$$

$$k_3 = a(1 + \zeta + \zeta^2)^2;$$

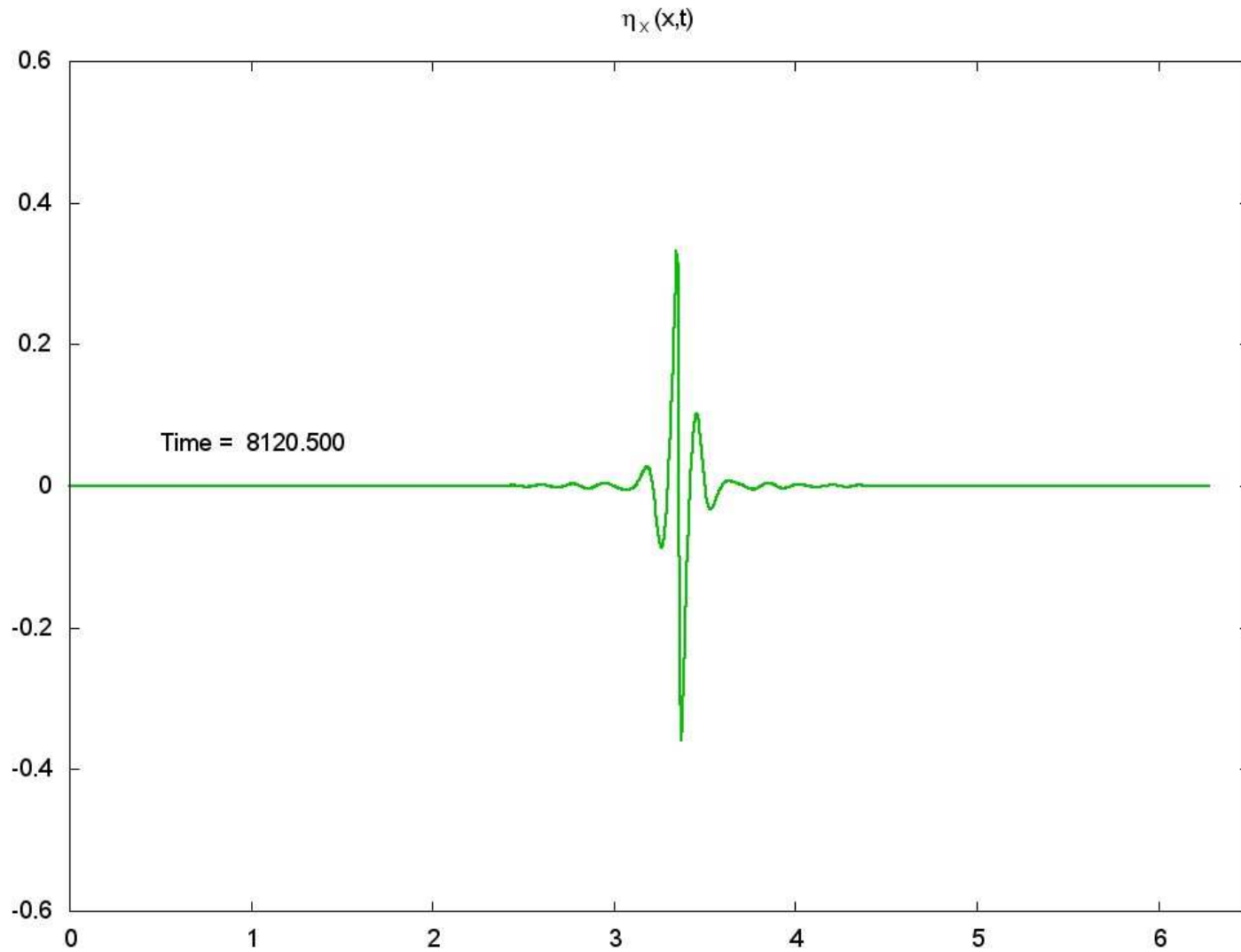
here  $0 < \zeta < 1$  and  $a > 0$ .

$$T_{kk_1}^{k_2k_3} = 0!$$

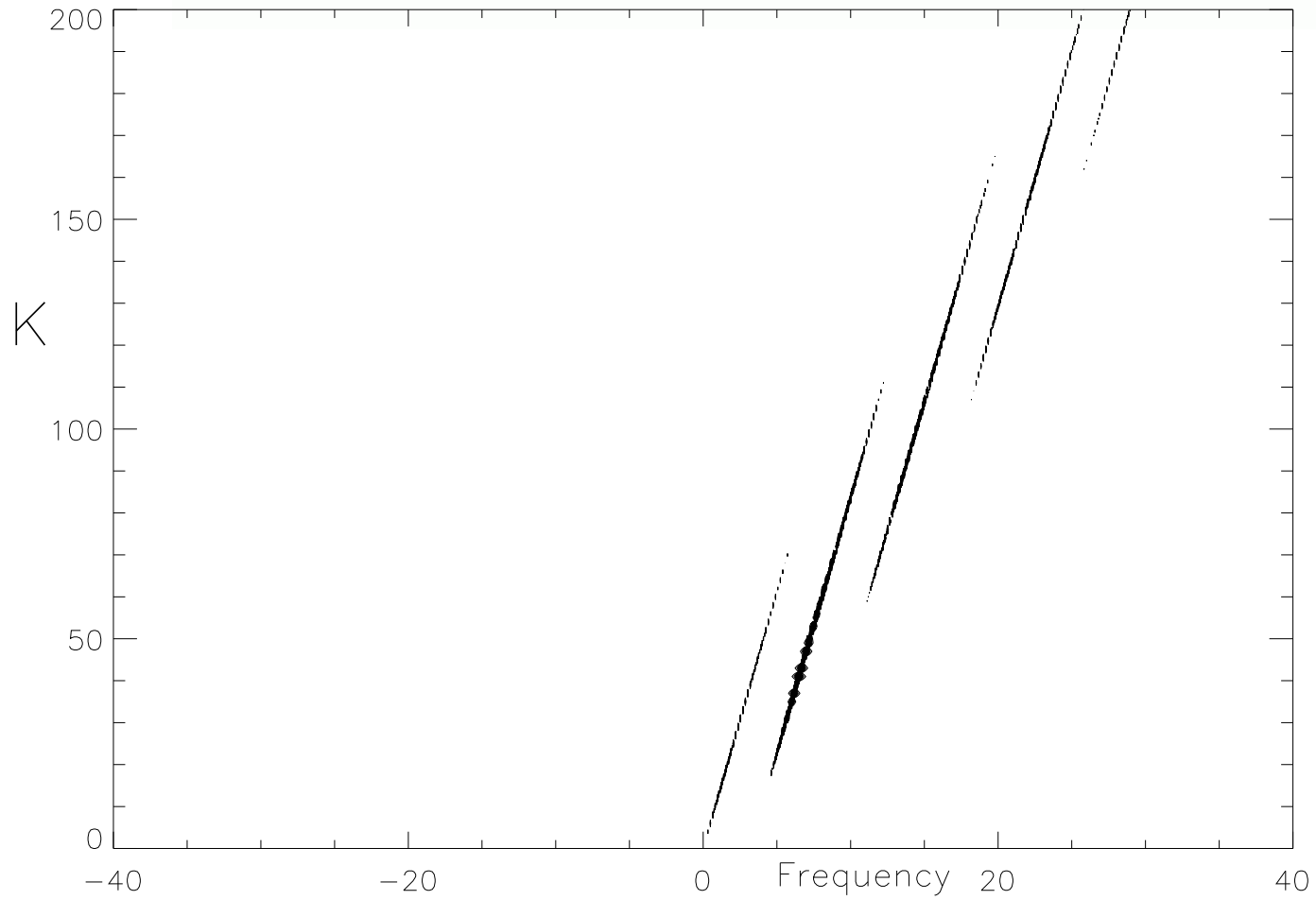
If so, it is possible to exclude four-wave interactions

?

# *Giant breather*



# *$k$ - $\omega$ spectrum of giant breather*



## Fact and observation

- ⑥ Waves moving in the same direction preserve this property (there is no backward radiation). So we can consider waves with  $k > 0$ .
- ⑥ Nontrivial 4-wave resonance is absent. So, only wave scattering to itself is important.

$$k + k_1 = k + k_1,$$

$$\omega_k + \omega_{k_1} = \omega_k + \omega_{k_1},$$

This scattering is described by diagonal part of  $T_{kk_1}^{k_2k_3}$ , which is

$$T_{kk_1}^{kk_1} = \hat{T}_{kk_1} = \frac{1}{4\pi} |k| |k_1| (|k + k_1| - |k - k_1|).$$

# Choice of canonical transformation

Using this diagonal part ( $\hat{T}_{kk_1}$ ), one can construct the following function:

$$\tilde{T}_{k_2 k_3}^{kk_1} = \left[ \frac{1}{2}(T_{kk_2} + T_{kk_3} + T_{k_1 k_2} + T_{k_1 k_3}) - \frac{1}{4}(T_{kk} + T_{k_1 k_1} + T_{k_2 k_2} + T_{k_3 k_3}) \right] \theta(kk_1 k_2 k_3)$$

$\tilde{T}_{kk_1}^{kk_1}$  coincides with original four-wave coefficient on the resonant manifold. Choose  $\tilde{B}_{k_2 k_3}^{kk_1}$  as follow

$$\tilde{B}_{k_2 k_3}^{kk_1} = \frac{\tilde{T}_{k_2 k_3}^{kk_1} - T_{k_2 k_3}^{kk_1}}{\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}}$$

$$\mathcal{H} = \int \omega_k b_k b_k^* dk + \frac{1}{4} \int \tilde{T}_{kk_1}^{k_2 k_3} b_k^* b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3 + \dots$$

# Compact Hamiltonian

$$\mathcal{H} = \int b^* (g\hat{K})^{1/2} b dx + \frac{1}{4} \int |b'|^2 \left[ \frac{i}{2} (bb'^* - b^*b') - \hat{K}|b|^2 \right] dx.$$

Corresponding dynamical equation is

$$i \frac{\partial b}{\partial t} = \hat{(gK)}^{1/2} b + \frac{i}{8} \left[ b^* \frac{\partial}{\partial x} (b'^2) - \frac{\partial}{\partial x} (b^{*'} \frac{\partial}{\partial x} b^2) \right] - \frac{1}{4} \left[ b \cdot \hat{K} (|b'|^2) - \frac{\partial}{\partial x} (b' \hat{K} (|b|^2)) \right].$$

# Monochromatic wave

$$b(x) = B_0 e^{i(k_0 x - \omega_0 t)}, \quad B_0 \text{-is arbitrary}$$

is the simplest solution. One can get the following relation

$$\omega_0 = \omega_{k_0} + \frac{1}{2} k_0^3 |B_0|^2.$$

$$|B_0|^2 = \frac{\omega_{k_0}}{k_0} \eta_0^2,$$

One can recover well known Stokes correction to the frequency due to finite wave amplitude.

$$\omega_0 = \omega_{k_0} \left( 1 + \frac{1}{2} k_0^2 |\eta_0|^2 \right).$$



# NLSE, Dysthe and beyond

Equation for amplitude of the wave train can be also easily derived. Let us introduce envelope  $B(x, t)$  so that

$$b(x, t) = B(x, t)e^{i(k_0x - \omega_0t)}.$$

$B(x, t)$  is also normal hamiltonian variable and Hamiltonian for it is the following:

$$\begin{aligned} \mathcal{H} &= \int B^* (\hat{\omega}_{k_0+k} - \omega_{k_0}) B dx + \\ &+ \frac{1}{4} \int |B' + ik_0 B|^2 \left[ \frac{i}{2} (BB'^* - B^* B') + k_0 |B|^2 - \hat{K} |B|^2 \right] dx \end{aligned}$$

# NLSE, Dysthe and beyond

$B(x, t)$  can be slowly varying function. Then expand  $(\hat{\omega}_{k_0+k} - \omega_{k_0})$  and neglect derivatives in nonlinear term one can get NLSE

$$i\dot{B} = -i\omega'_{k_0} B' - \frac{1}{2}\omega''_{k_0} B'' + \frac{1}{2}k_0^3 |B|^2 B.$$

If we keep first derivatives in the Hamiltonian we derive Dysthe equation.

$$\begin{aligned} \mathcal{H} &= \int B^* (\hat{\omega}_{k_0+k} - \omega_{k_0}) B dx + \\ &+ \frac{1}{4} \int k_0^2 |B|^2 \left[ \frac{3i}{2} (BB'^* - B^* B') + k_0 |B|^2 - \hat{K} |B|^2 \right] dx. \end{aligned}$$

# Modulational instability of monochromatic wave

Consider solution as follow:

$$b = (B_0 + \delta b(x, t))e^{i(k_0x - \omega_0t)}$$

where  $B_0 = \text{const.}$  Linearized equation for  $b(x, t)$  has solution as follow

$$\delta b \Rightarrow \delta b e^{\gamma_k t + i(kx - \Omega_k t)},$$

Then for growth rate  $\gamma_k$  the following formula is valid:

$$\gamma_k^2 = \frac{1}{8} \frac{\omega_{k_0}^2}{k_0^4} (1 - 6\mu^2) k^2 \left[ \mu^2 \left( k_0 - \frac{|\mathbf{k}|}{2} \right)^2 - \frac{k^2}{8} \right].$$

# Conclusions

$$\mathcal{H} = \int b^* \hat{\omega}_k b dx + \frac{1}{4} \int |b'|^2 \left[ \frac{i}{2} (bb'^* - b^*b') - \hat{K} |b|^2 \right] dx.$$

In the spirit of Kadomtsev-Petviashvili equation for Korteweg-de-Vries equation:

$$\mathcal{H} = \int b^* \hat{\omega}_{k_x, k_y} b dx dy + \frac{1}{4} \int |b'_x|^2 \left[ \frac{i}{2} (bb'_x{}^* - b^*b'_x) - \hat{K}_x |b|^2 \right] dx dy.$$