Soliton on Unstable Condensate

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Focusing Nonlinear Schrödinger equation (NLSE)

We study solutions of the following NLSE:

$$i\varphi_t - \frac{1}{2}\varphi_{xx} - (|\varphi|^2 - A^2)\varphi = 0 \tag{1}$$

with boundary conditions $|\varphi|^2 \to A^2$ at $x \to \pm\infty$. Here $A = \overline{A}$ is a real constant.

Applications

Bose-Einstein condensate (E.P. Gross and L.P. Pitaevskii, 1961)

Nonlinear optics (R. Y. Chiao, E. Garmire, and C. H. Townes, 1964)

Ocean waves (V.E. Zakharov, 1968)

It has been known since 1971 that the focusing NLSE is a very special system, integrable by the Inverse Scattering Method (V.E. Zakharov and A.B. Shabat)

Particular solutions

E.A. Kuznetsov, 1977
this solution was rediscovered by other authors:
T. Kawata and H. Inoue, 1978
Y-C Ma, 1979

Peregrine, 1983

N.N. Akhmediev and V.I. Korneev, 1986

Lax system

NLSE equation (1) is the compatibility condition for the following overdetermined linear system for a matrix function Ψ:

$$\frac{\partial \Psi}{\partial x} = \widehat{U}\Psi, \quad i\frac{\partial \Psi}{\partial t} = (\lambda \widehat{U} + \widehat{W})\Psi \tag{2}$$

Here:

$$\widehat{U} = I\lambda + u$$
$$I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u = \begin{pmatrix} 0 & \varphi \\ -\overline{\varphi} & 0 \end{pmatrix}$$
$$\widehat{W} = \frac{1}{2}\begin{pmatrix} |\varphi|^2 - |A|^2 & \varphi_x \\ \overline{\varphi}_x & -|\varphi|^2 + |A|^2 \end{pmatrix} \tag{3}$$

Condensate solution

If $\varphi = A$, system (2) has the following solution:

$$\Psi_0 = \frac{1}{\sqrt{1-q^2}} \begin{pmatrix} e^{\phi} & q \cdot e^{-\phi} \\ q \cdot e^{\phi} & e^{-\phi} \end{pmatrix} \tag{4}$$

Here:

$$\phi = kx + \Omega t, \quad k^2 = \lambda^2 - A^2$$
$$\Omega = -i\lambda k, \quad q = -\frac{A}{\lambda + k}$$

Note that:

$$\overline{k}(-\overline{\lambda}) = -k(\lambda), \quad \overline{q}(-\overline{\lambda}) = -q(\lambda), \quad \overline{\phi}(-\overline{\lambda}) = \phi(\lambda)$$

Local matrix $\overline{\partial}$ problem

We consider the $\overline{\partial}$-problem on the complex λ-plane. We are looking for a second order matrix $\chi(\lambda, \overline{\lambda}, x, t)$ obeying the equation:

$$\frac{\partial \chi}{\partial \overline{\lambda}} = \chi \cdot f(\lambda, \overline{\lambda}, x, t) \qquad (5)$$

and normalized by condition $\chi \to 1$ at $|\lambda| \to \infty$. Here:

$$f = \psi_0 f_0(\lambda, \overline{\lambda}) \psi_0^{-1}$$

$$f_0(\lambda, \overline{\lambda}) = f_0^{+}(\lambda, \overline{\lambda}) \qquad (6)$$

f_0 is a "dressing function"

Local matrix $\bar{\partial}$ problem

The function χ has an asymptotic expansion at $\lambda \to \infty$

$$\chi = 1 + \frac{R}{\lambda} + \cdots \tag{7}$$

The function χ satisfies the following system of equations:

$$\begin{aligned} \frac{\partial \chi}{\partial x} &= \widehat{U}\chi - \chi\widehat{U_0} \\ i\frac{\partial \chi}{\partial t} &= (\lambda\widehat{U} + \widehat{W}) - \lambda\chi\widehat{U_0} \end{aligned} \tag{8}$$

Because this system (8) is overdeterminated, we have the following expression for φ:

$$\varphi = A - 2R_{(12)} \tag{9}$$

"Dressing function"

Let us choose:

$$f_0(\lambda,\overline{\lambda}) = \begin{pmatrix} 0 & \alpha(\lambda) \\ \overline{\alpha}(-\overline{\lambda}) & 0 \end{pmatrix}$$

The function f now takes the form:

$$f(\lambda,\overline{\lambda},x,t) = \alpha e^{2\phi} A + \overline{\alpha}(-\overline{\lambda},-\lambda) B e^{-2\phi}$$

The matrices A, B are degenerate:

$$A_{\alpha\beta} = a_\alpha b_\beta, \quad B_{\alpha\beta} = c_\alpha d_\beta$$

$$a = (1,q), \quad b = (-q,1); \qquad c = (q,1), \quad d = (1,-q)$$

"Dressing function"

We now choose:

$$\alpha(\lambda, \overline{\lambda}) = C\delta(\lambda - \eta)$$

We will find a solution of the $\overline{\partial}$ - problem (5) in the form of rational functions with two poles:

$$\chi = 1 + \frac{U}{\lambda - \eta} + \frac{V}{\lambda + \overline{\eta}} \quad (10)$$

where U, V are constant degenerate matrices:

$$U_{\alpha\beta} = u_\alpha b_\beta, \quad V_{\alpha\beta} = v_\alpha a^*_\beta$$

Substituting (10) into (5):

$$\pi\{U\delta(\lambda-\eta)+V\delta(\lambda+\eta^*)\} = \{1+\frac{U}{\lambda-\eta}+\frac{V}{\lambda+\eta^*}\}\{\widetilde{A}\delta(\lambda-\eta)+\widetilde{B}\delta(\lambda+\eta^*)\}$$

we end up with a simple linear system of equations for u_α and v_α:

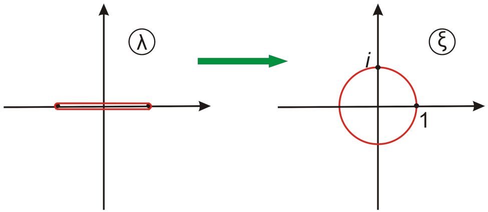
$$u_\alpha(1+\frac{q}{k}Ce^{2\phi}) - \frac{1+|q|^2}{\eta+\overline{\eta}}Ce^{2\phi}v_\alpha = a_\alpha Ce^{2\phi}$$
$$\frac{1+|q|^2}{\eta+\overline{\eta}}Ce^{2\overline{\phi}}u_\alpha + (1+\frac{\overline{q}}{\overline{k}}\overline{C}e^{2\overline{\phi}})v_\alpha = \overline{b}_\alpha\overline{C}e^{2\overline{\phi}}$$

By virtue of 9 the solution of equation (1) is given as follows:

$$\varphi = A - 2(u_1 + v_1\overline{q})$$

Uniformizing variable

$$\lambda = \frac{A}{2}(\xi + \frac{1}{\xi}), \quad k = \frac{A}{2}(\xi - \frac{1}{\xi})$$



$$\phi = \frac{1}{2}(æx - \gamma t) + i\frac{1}{2}(kx - \omega t)$$

$$k = A(R + \frac{1}{R})\sin(\alpha), \quad \omega = \frac{A^2}{2}(R^2 - \frac{1}{R^2})\cos(2\alpha)$$

$$æ = A(R - \frac{1}{R})\cos(\alpha), \gamma = -\frac{A^2}{2}(R^2 + \frac{1}{R^2})\sin(2\alpha)$$

General solution

$$\varphi = \frac{Ae^{2i\alpha}}{2}\left(\frac{2\cos(2\alpha)\cosh(u+w) - \frac{1}{a}(R^2+\frac{1}{R^2})\cos(v)}{\cosh(u+w) - \frac{1}{a}\cos(v)} + i\frac{2\sin(2\alpha)\sinh(u+w) + (R^2-\frac{1}{R^2})\sin(v)}{\cosh(u+w) - \frac{1}{a}\cos(v)}\right)$$

Here:

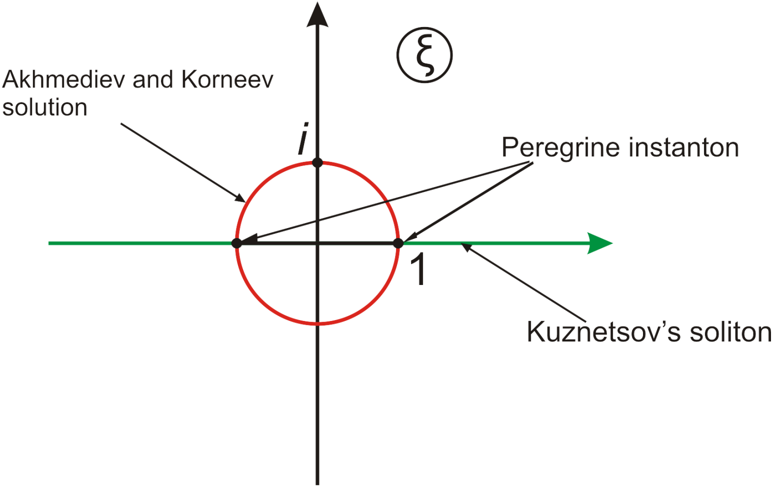
$$u = \phi + \phi^*, \quad v = \phi - \phi^*$$

$$a = \frac{1+R^2}{2R\cos(\alpha)}, \quad w = \ln(a)$$

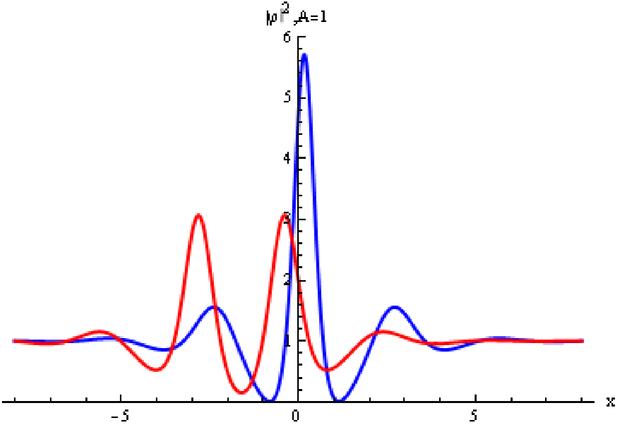
Asymptotic behavior

$$\varphi \to A \quad \text{at} \quad x \to -\infty$$
$$\varphi \to Ae^{4i\alpha} \quad \text{at} \quad x \to +\infty$$
$$|\varphi|^2 = A^2 \quad \text{at} \quad x \to \pm\infty$$

Particular solutions

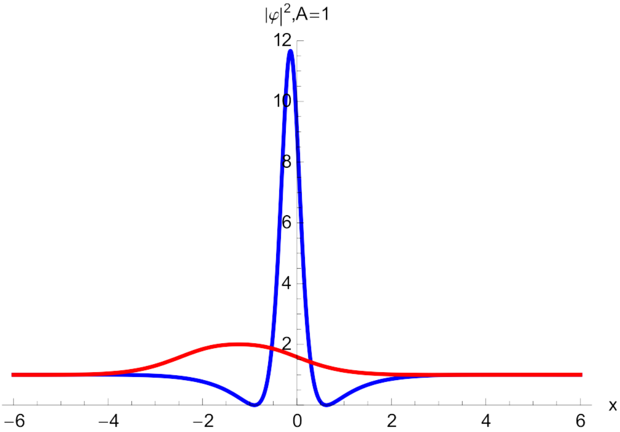


Typical solution



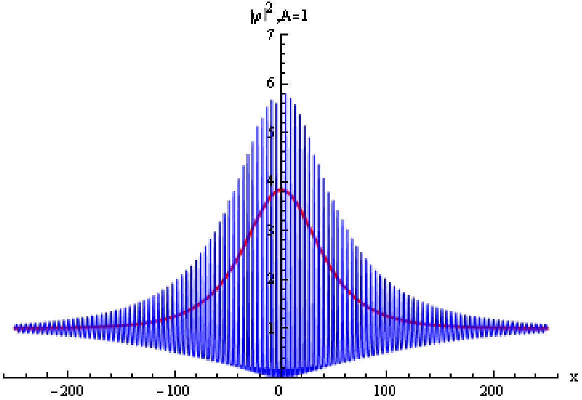
Typical solitonic solution at the moment of maximum (blue) and minimum (red) of amplitude. ($R = 2, \alpha = \frac{5}{16}\pi$)

Kuznetsov-like solution



Kuznetsov-like solution at the moment of maximum (blue) and minimum (red) of amplitude ($R = 2, \alpha = \frac{\pi}{12}$).

Near Akhmediev and Korneev case



Soliton and its envelope. ($R = 1.01, \alpha = \frac{\pi}{4}$)

Thank you for your attention!