Vortex line representation and breaking of vortex field lines in hydrodynamics

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OUTLINE

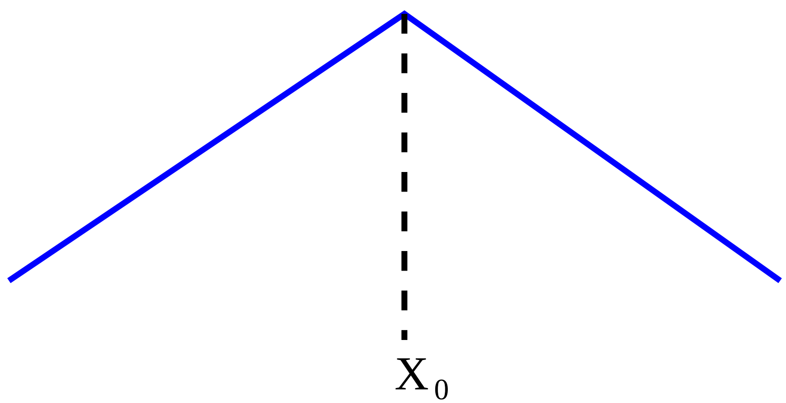
- Motivation: collapses and turbulent spectra
- Relabeling symmetry and Cauchy invariant
- Vortex line representation and new Cauchy invariant
- Breaking of vortex lines
- Numerical experiment

Collapses and turbulent spectra

It is well known that singularities give the power type behavior of the Fourier amplitudes that provides appearance of power tails for turbulent spectra.

Examples:

- (1958) Phillips spectrum for gravity waves on the fluid surface. Surface singularities are wedges:



$$\Leftarrow \quad z = \eta(x,t) \Rightarrow$$

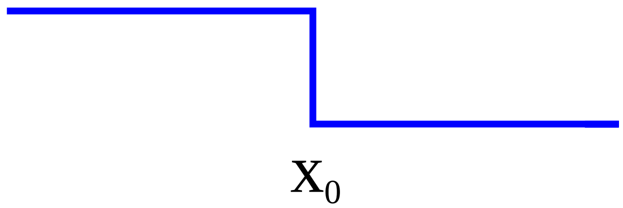
$\eta_{xx} \sim \delta(x - x_0)$

or $\eta_k \sim k^{-2}$. Hence one can get the Phillips spectrum:

$\varepsilon_\omega \sim \omega^{-5}$, $\omega = \sqrt{gk}$.

(Kuznetsov, 2004):$\varepsilon_k \sim k^{-4}$ for isotropic case and $\varepsilon_\omega \sim \omega^{-4}$ for frequency spectrum.

- (1973) Kadomtsev-Petviashvili spectrum. According to KP acoustic turbulence is a random set of shocks:



$$\rho_x \sim \delta(x - x_0),$$
$$\rho_k \sim k^{-1} \;\Rightarrow$$
$$\varepsilon_\omega \sim \omega^{-2}.$$

- (1941) Kolmogorov spectrum, i.e. the energy distribution of the velocity fluctuations in the inertial interval ($Re \gg 1$),

  $E_k \sim P^{2/3} k^{-5/3}$

  where P is the energy flux. This spectrum can be obtained from the dimensional analysis.

Collapses and turbulent spectra

- Using this analysis one can get that the energy transfer time T from large scales L to dissipative ones is finite and defined by L and P :
  $T \sim L^{2/3}P^{-1/3}$.
- Distribution of velocity fluctuations
  $v \sim P^{1/3}r^{1/3}$
  Respectively, for vorticity $\Omega = \text{curl}\,\mathbf{v}$ we have:
  $$\Omega \sim P^{1/3}r^{-2/3}.$$
  Thus, for Ω we have singularity at $r \to 0$, besides T is finite.
  Question: Is it a real singularity?

Temporal behavior of vorticity at the collapse point - naive arguments

The Euler equation for vorticity $\boldsymbol{\Omega}$

$$\frac{d\boldsymbol{\Omega}}{dt} = (\boldsymbol{\Omega}\cdot\nabla)\mathbf{v}, \quad \frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla), \quad \text{div}\,\mathbf{v} = 0.$$

At the maximal point of vorticity

$$\frac{d\Omega_{max}}{dt} = \frac{\partial v_\tau}{\partial x_\tau}\Omega_{max}, \quad \text{where } \tau = \frac{\boldsymbol{\Omega}}{|\boldsymbol{\Omega}|}.$$

If $\frac{\partial v_\tau}{\partial x_\tau} = \alpha\Omega_{max}$ then we have the ODE, $\frac{d\Omega_{max}}{dt} = \alpha\,\Omega_{max}^2$, with the blow-up solution: $\Omega_{max} \sim (t_0 - t)^{-1}$.

From another side, if one assumes that $\Omega_{max} \approx c(t_0 - t)^{-1}$ then

$$\frac{\partial v_\tau}{\partial x_\tau} = \frac{1}{t_0 - t}$$

that can be considered as a checking rule for exact solution.

Relabeling symmetry and Cauchy invariant

It is well known that the Euler equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\nabla p, \qquad \text{div}\ \mathbf{v} = 0,$$

has infinite (continuous) number of integrals of motion. These are the so called Cauchy invariants. They can be obtained from the Kelvin theorem

$$\Gamma = \oint_{C[t]} (\mathbf{v} \cdot d\mathbf{l}) = \text{inv}$$

with the movable together with fluid contour $C[t]$. Passing in this integral to the Lagrangian variables,

$$\mathbf{r} = \mathbf{r}(\mathbf{a}, t), \ \frac{d\mathbf{r}}{dt} = \mathbf{v}(\mathbf{r}, t), \ \mathbf{r}|_{t=0} = \mathbf{a}$$

we arrive at

$$\Gamma = \oint_{C[a]} \dot{x}_i \cdot \frac{\partial x_i}{\partial a_k}\, da_k \ , \ \text{with fixed } C[a].$$

Relabeling symmetry and Cauchy invariant

Hence we get the Cauchy invariants

$$\mathbf{I} = \text{curl}_a \left(\dot{x}_i \frac{\partial x_i}{\partial \mathbf{a}} \right) \equiv \mathbf{\Omega_0}(\mathbf{a})$$

which are constraints in Euler. Conservation of the Cauchy invariants is a consequence of the relabeling symmetry (R. Salmon, 1982). They characterize the frozenness of the vorticity into fluid. The latter means that fluid (Lagrangian) particles can not leave its own vortex line where they were initially. Thus, the Lagrangian particles have one independent degree of freedom – motion along vortex line. But such a motion does not change the vorticity:

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \text{curl}\, [\mathbf{v} \times \mathbf{\Omega}].$$

Vortex line representation

Thus, the Helmholtz equation contains only one velocity component normal to the vortex line, $\mathbf{v_n}$. The tangent velocity $\mathbf{v}_\tau$ plays a passive role providing incompressibility. Decomposing,$\mathbf{v} = \mathbf{v_n} + \mathbf{v}_\tau$, in the Euler *incompressible* equations leads to the the equation of motion of charged *compressible* fluid moving in an electromagnetic field:

$$\frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n \nabla)\mathbf{v}_n = \mathbf{E} + [\mathbf{v}_n \times \mathbf{H}],$$

where

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{H} = \text{curl } \mathbf{A}$$

with $\varphi = p + v_\tau^2/2, \ \mathbf{A} = \mathbf{v}_\tau$. Thus, two Maxwell equations are satisfied with the gauge: $\text{div}\,\mathbf{A} = -\text{div}\,\mathbf{v}_n \neq 0.$

Vortex line representation

Now perform transform in a new charged *compressible* hydrodynamics to the Lagrangian description:

$$\dot{\mathbf{r}} = \mathbf{v}_n(\mathbf{r}, t) \text{ with } \mathbf{r}|_{t=0} = \mathbf{a}.$$

Under this transform the new hydrodynamics become the Hamilton equations:

$$\dot{\mathbf{P}} = -\partial h/\partial \mathbf{r}, \qquad \dot{\mathbf{r}} = \partial h/\partial \mathbf{P},$$

$\mathbf{P} = \mathbf{v}_n + \mathbf{A} \equiv \mathbf{v}$ is the generalized momentum, and the Hamiltonian $h = (\mathbf{P} - \mathbf{A})^2/2 + \varphi \equiv p + \mathbf{v}^2/2$ ($\equiv$ the Bernoulli "invariant").

The Kelvin (Liouville) theorem says that $\Gamma = \oint (\mathbf{P} \cdot d\mathbf{R}) = \text{inv}$.

Transform in Γ to new Lagrangian coordinates leads to a new Cauchy invariant :

$$\mathbf{I} = \text{curl}_a \left(P_i \frac{\partial x_i}{\partial \mathbf{a}} \right) \equiv \mathbf{\Omega_0}(\mathbf{a}).$$

Vortex line representation (VLR)

Hence we get

$$\boldsymbol{\Omega}(\mathbf{r},t) = \frac{(\boldsymbol{\Omega}_0(\mathbf{a}) \cdot \nabla_a)\mathbf{r}(\mathbf{a},t)}{J(\mathbf{a},t)}$$

where $\mathbf{P}$ coincides with $\mathbf{v}$ and $J(\mathbf{a},t) = \partial(\mathbf{r})/\partial(\mathbf{a})$ is the Jacobian of the mapping $\mathbf{r} = \mathbf{r}(\mathbf{a},t)$ defined from

$$\dot{\mathbf{r}} = \mathbf{v}_n(\mathbf{r},t),\ \mathbf{r}|_{t=0} = \mathbf{a}.$$

These equations together with

$$\boldsymbol{\Omega}(\mathbf{r},t) = \text{curl}_r\, \mathbf{v}(\mathbf{r},t) \text{ and } \text{div}_r\, \mathbf{v}(\mathbf{r},t) = 0$$

form the complete system of equations in the vortex line representation (Kuznetsov, Ruban (1998), Kuznetsov (2002, 2006)).

The quantity $n = J^{-1}$ plays the role of vortex line density:

$$n_t + \text{div}_r(n\mathbf{v}_n) = 0,\ \ \text{div}_r \mathbf{v}_n \neq 0.$$

Clebsch variables

Due to frozenness, enumerate each vortex line by 2D Lagrangian label ν with introducing the parameter s given the vortex line. Clebsch variables can be used as such Lagrangian markers: $\mathbf{\Omega} = [\nabla\lambda \times \nabla\mu]$. Intersection of $\lambda = \text{const}$ and $\mu = \text{const} \to$ the vortex line. Besides, $\dot{\lambda} = \dot{\mu} = 0$. By using the change of variables,

$\lambda = \lambda(x, y, z), \qquad \mu = \mu(x, y, z), \qquad s = s(x, y, z),$

we arrive at

$$\mathbf{\Omega}(\mathbf{r}, t) = \frac{1}{J} \cdot \frac{\partial \mathbf{r}}{\partial s}, \quad J = \frac{\partial(x, y, z)}{\partial(\lambda, \mu, s)}.$$

Eq. of motion is the same: $\dot{\mathbf{r}} = \mathbf{v_n}(\mathbf{r}, t)$.

Therefore the VLR represents the mixed Lagrangian-Eulerian description.

Another applications of VLR

VLR can be applied also to the whole family of equations:

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \text{curl}\,[\mathbf{v} \times \mathbf{\Omega}] = \{\mathbf{\Omega}, \mathcal{H}\}, \quad \mathbf{v} = \text{curl}\,\frac{\delta \mathcal{H}}{\delta \mathbf{\Omega}}.$$

The VLR formula reads the same:

$$\mathbf{\Omega}(\mathbf{r}, t) = \frac{(\mathbf{\Omega}_0(\mathbf{a}) \cdot \nabla_a)\mathbf{r}(\mathbf{a}, t)}{J(\mathbf{a}, t)}$$

where

$$\left[\tau \times \left(\frac{\partial \mathbf{r}}{\partial t} - \mathbf{v}(\mathbf{r}, t)\right)\right] = 0.$$

and

$\tau = \mathbf{\Omega}/|\mathbf{\Omega}|$ is the unit tangent vector.

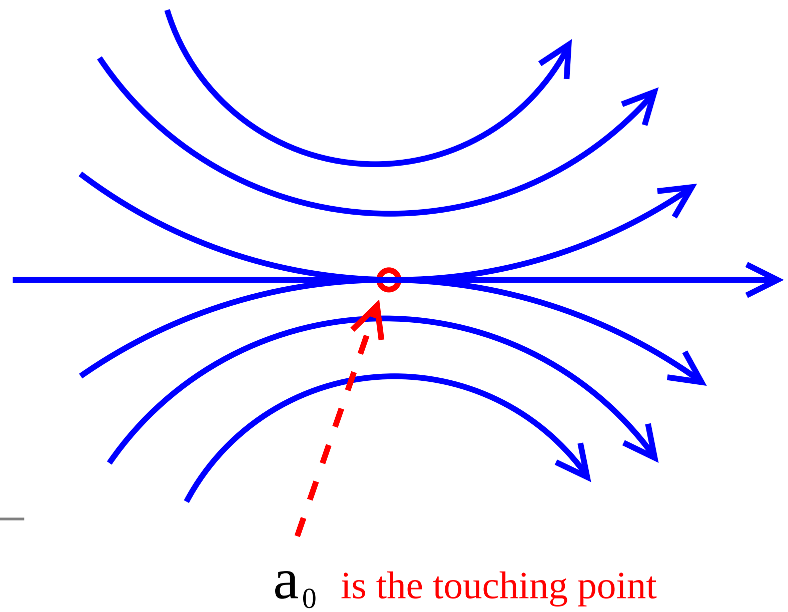
Breaking of vortex lines

REMARK 1: Breaking in gasdynamics corresponds to vanishing J. Because of compressible character of the mapping we can expect breaking of vortex lines.

REMARK 2: Breaking of vortex lines is impossible in 2D and for cylindrically symmetric flows without swirl (Majda, 1990) because $\mathbf{\Omega} \perp \mathbf{v}$ and div $\mathbf{v}_n = 0$, and consequently $J = 1$.

Thus, breaking of vortex line is 3D phenomenon.

Geometrically this results in touching of vortex lines.

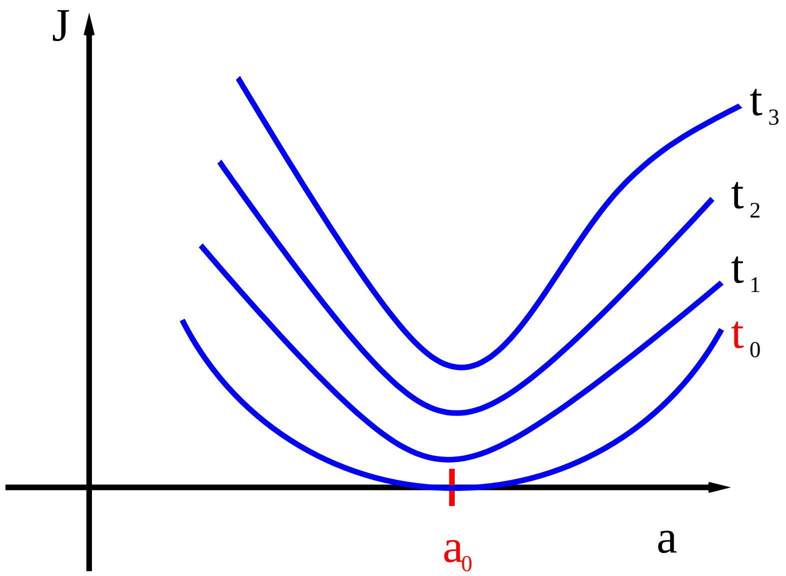


Breaking of vortex lines

Let us assume that breaking takes place. Consider the equation $J(\mathbf{a}, t) = 0$ and find its positive roots $t = \tilde{t}(\mathbf{a}) > 0$. Then the collapse (or touching) time will be

$$t_0 = \min_a \tilde{t}(\mathbf{a}).$$

Near the minimal point $\mathbf{a} = \mathbf{a}_0$ the expansion of J takes



the form:

$J(a, t) = \alpha(t_0 - t) + \gamma_{ij}\Delta a_i \Delta a_j$

- concavity condition

$\alpha > 0$,

γ_{ij} is positive definite (nongenerated) matrix,

$\Delta\mathbf{a} = \mathbf{a} - \mathbf{a}_0$.

Breaking of vortex lines

This expansion results in self-similar asymptotics for vorticity:

$$\mathbf{\Omega}(\mathbf{r},t) = \frac{(\mathbf{\Omega_0}(\mathbf{a}) \cdot \nabla_a)\mathbf{r}|_{a_0}}{\tau(\alpha + \gamma_{ij}\eta_i\eta_j)}, \;\; \eta = \Delta a, \;\; \nu = (\lambda, \mu), \;\; \tau = t_0 - t.$$

NOTE: This self-similar asymptotics is marginal in the sense of the Beale-Kato-Majda criterion:

$$\int_0^{t_0} \max_r |\mathbf{\Omega}| dt \;\; = \left\{ \begin{array}{c} \infty \text{ for collapsing solutions} \\ < \infty \text{ for noncollapsing solutions} \end{array} \right\}$$

Now the main problem is

to transform from the auxiliary a-space to the physical $\mathbf{r}$-space.

Structure of singularity

1D case

$$v_t + vv_x = 0 \ \text{(dust)}, \ \ p = 0.$$

Solution is given in the implicit form:

$$v = v_0(a), \ \ x = a + v_0(a)t.$$

Breaking (or gradient catastrophe, or formation of fold) results in the infinite density and velocity gradient ($n = n_0/J$, $\partial v/\partial x = v_o'(a)/J$):

$$J = \frac{\partial x}{\partial a} = \alpha\tau + \gamma a^2 \ \ \rightarrow x = \alpha\tau a + \frac{1}{3}\gamma a^3.$$

Thus, $a \sim \tau^{1/2}, \ \ x \sim \tau^{3/2}$!! At $\tau = 0$ we have the singularity:

$$n \sim \frac{\partial v}{\partial x} \sim x^{-2/3}.$$

Structure of singularity

3D case

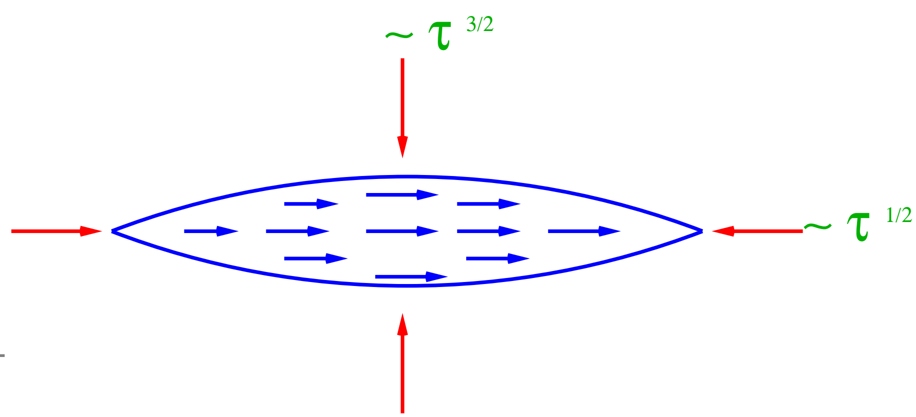
The Jacobian $J = \lambda_1\lambda_2\lambda_3 \to 0$ means that one eigenvalue, say, $\lambda_1 \to 0$ and $\lambda_2, \lambda_3 \to$ const as $t \to t_0$ and $a \to a_0$. Hence it follows that near singular point there are two different self similarities:

along "soft" (λ_1) direction $x_1 \sim \tau^{3/2}$ (like in 1D);

along "hard" (λ_2, λ_3) directions $x_{2,3} \sim \tau^{1/2}$,

so that

$$\boldsymbol{\Omega} = \frac{1}{\tau}\mathbf{g}\left(\frac{x_1}{\tau^{3/2}}, \frac{x_\perp}{\tau^{1/2}}\right).$$



This results in formation of pancake structure (compare with Zeldovich)

Structure of singularity

At $\tau = 0$ we get a very anisotropic singularity. The main dependence of $\mathbf{\Omega}$ is connected with x_1-direction:

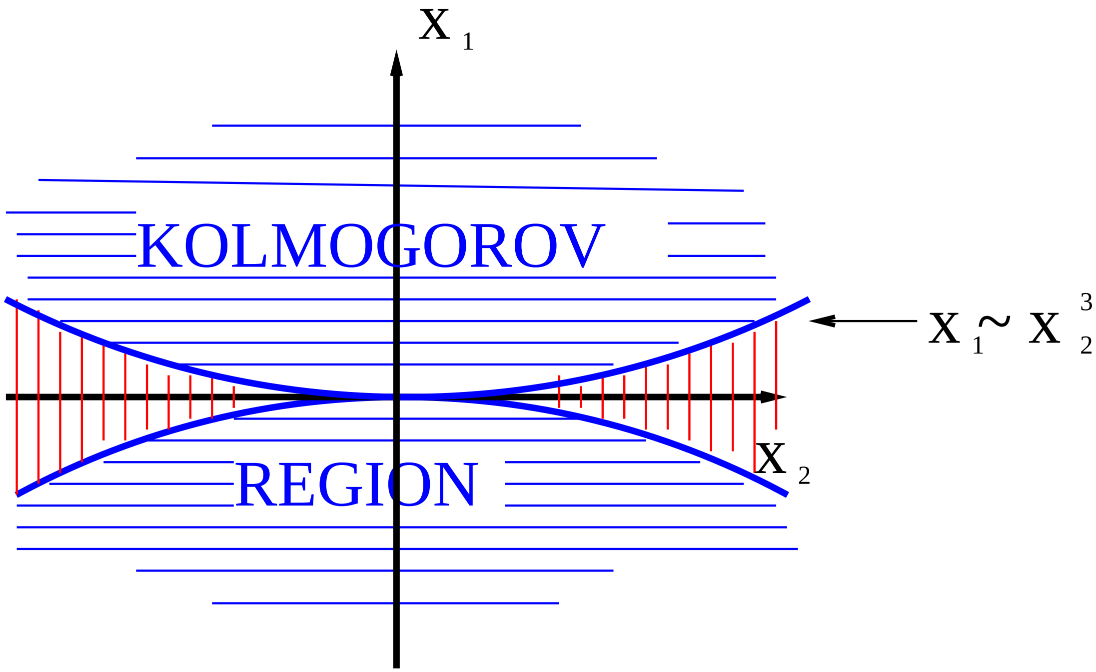
$$\mathbf{\Omega} \approx \frac{\mathbf{b}}{x_1^{2/3}}$$

with $\mathbf{b} =$ const and KOLMOGOROV index $2/3$!.

This dependence is realized everywhere except regions between two cubic paraboloids $-cx_\perp^3 < x_1 < cx_\perp^3$ In this narrow region vorticity at $\tau = 0$ behaves like

$$\mathbf{\Omega} \approx \frac{\mathbf{b_1}}{x_\perp^2}.$$

Structure of singularity



In Kolmogorov region the vorticity can be estimated as

$$\Omega \sim \frac{P^{1/3}}{x_1^{2/3}}$$

where $P \sim \Omega_0^3 L^2, \;\; L \sim \gamma^{-1/2}$.

Breaking of vortex lines in 3D integrable hydrodynamics

3D integrable hydrodynamics was introduced by Kuznetsov and Ruban (1998):

$$\mathcal{H} = \int |\mathbf{\Omega}(\mathbf{r})| d\mathbf{r} \rightarrow \text{ applying VLR } \rightarrow = \int d^2\nu \int \left| \frac{\partial \mathbf{R}}{\partial s} \right| ds$$

and, respectively, Eq. of motion is of the form:

$$[\mathbf{R}_s \times \mathbf{R}_t] = [\vec{\tau} \times [\vec{\tau} \times \vec{\tau}_s]]$$

where $\mathbf{r} = \mathbf{R}(\nu, s)$ are coordinates of the vortex loop ν. This equation $\rightarrow$ 1D Landau-Lifshitz or NLS (Hasimoto transform). This model can be obtained from 3D Euler by means of LIA.
This system represents ensemble of non-interacting continuously distributed free (!!) vortex lines. It is just the reason of existence of breaking of vortex lines in this model, in spite of both divergence-free velocity and vorticity.

Breaking of vortex lines in 3D integrable hydrodynamics

The simplest mapping can be constructed from the stationary solutions, $\mathbf{R}_t = \mathbf{V} \equiv \text{const}$, of the ring type:

$$\mathbf{V} = \kappa \mathbf{b}, \text{ or } V = 1/r$$

where $\kappa = r^{-1}$ is the curvature, and $\mathbf{b}$ the binormal. The mapping is

$$\mathbf{R} = \mathbf{R_0}(\nu) + r(\nu)\cos\phi \cdot \mathbf{e}_x + r(\nu)\sin\phi \cdot \mathbf{e}_y + V(\nu)t \cdot \mathbf{e}_z$$

with $\mathbf{e}_{x,y,z}$ being unit vectors. As the result, the Jacobian becomes a lunear function of time:

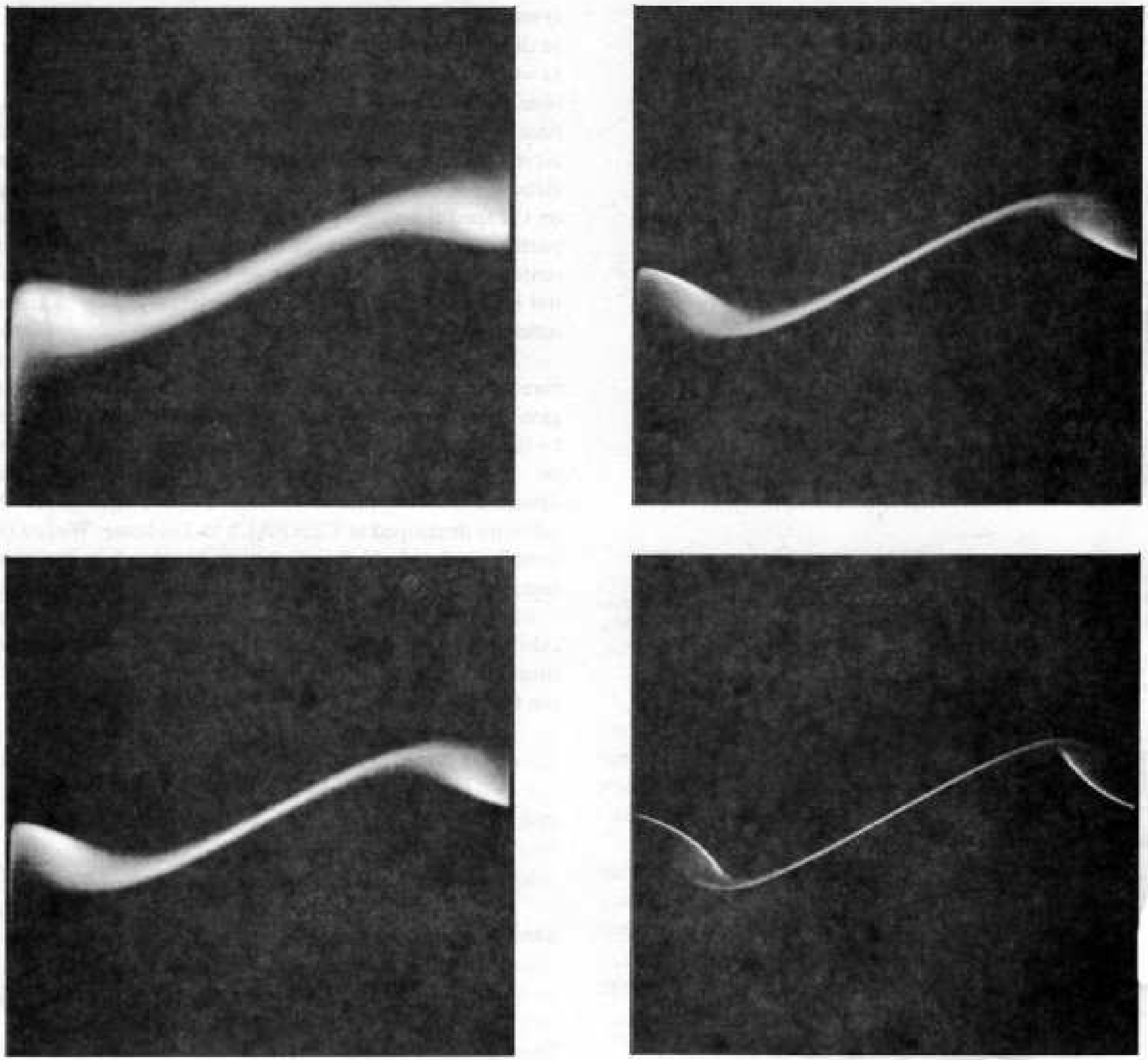
$$J = \frac{\partial(X,Y,Z)}{\partial(\lambda,\mu,s)} = J_0(\nu, s) + A(\nu, s)t.$$

Thus, we arrive here at the breaking of vortex lines (Kuznetsov, Ruban, 2000) familiar to that in gasdynamics.

Numerics

In the main numerics (Kerr (1993); Pelz (1997), Boratav & R.B. Pelz (1994), Grauer, Marliani, & Germaschewski (1998)) the vorticity maximum blows-up like $(t_0 - t)^{-1}$. Concerning the spatial structure, only a qualitative agreement takes place. The numerics (Brachet, Meneguzzi, Vincent, Politano, Sulem (1992)) showed for the initial conditions (the Taylor-Green vortex and random) the formation of thin vortex layers with high vorticity.
Before Rainer & Sideris (1991) also observed formation of pancake structures for axisymmetric flows with swirl.

Taylor-Green vortex, Brachet, et. al. (864^3)



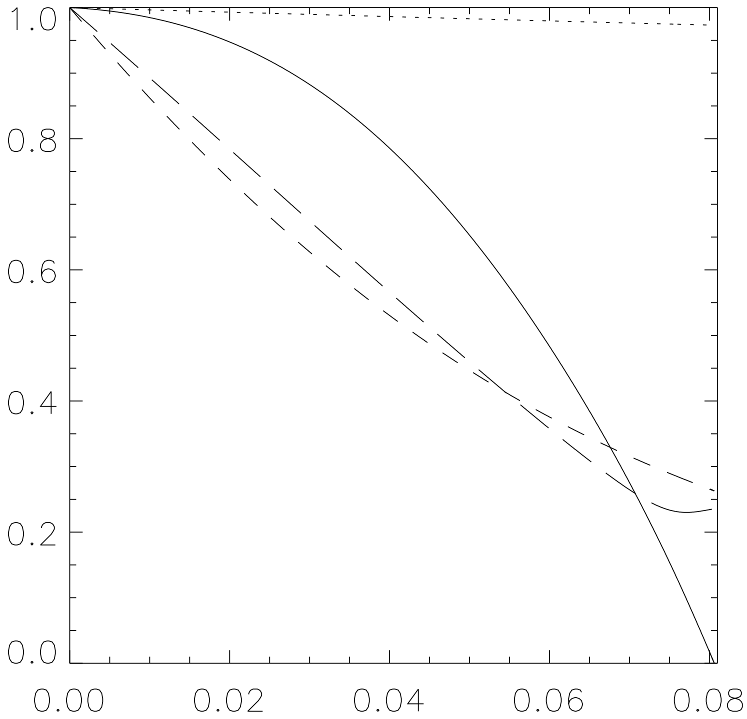
Visualizations in physical space of high vorticity regions for the Taylor–Green vortex at $t = 3$, 3.25, 3.50, and 4.

Numerics

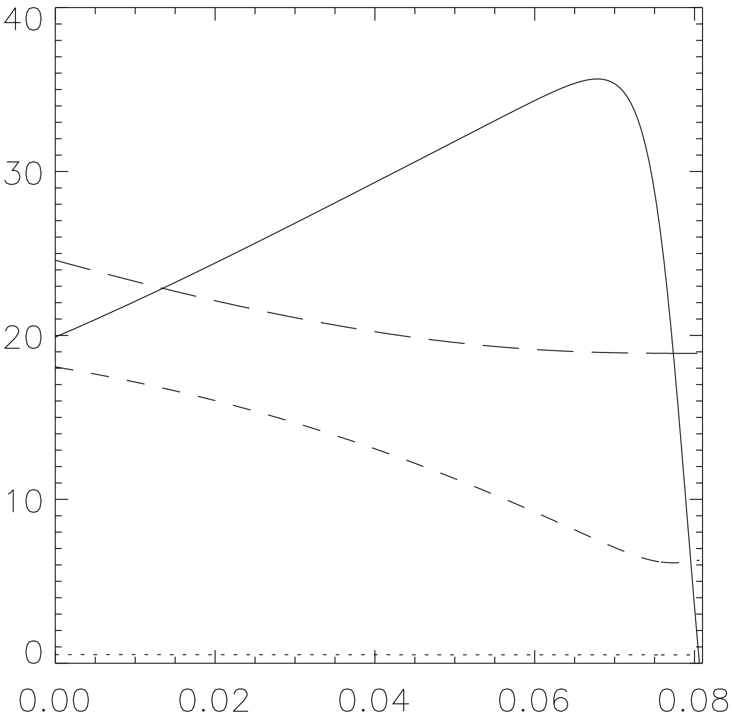
In our numerics (K., Zheligovsky & Podvigina (2001, 2002)) we demostrated that collapse took place due to vanishing of the Jacobian at one separate point (128^3 grids).

"Random" initial conditions

$\min J$

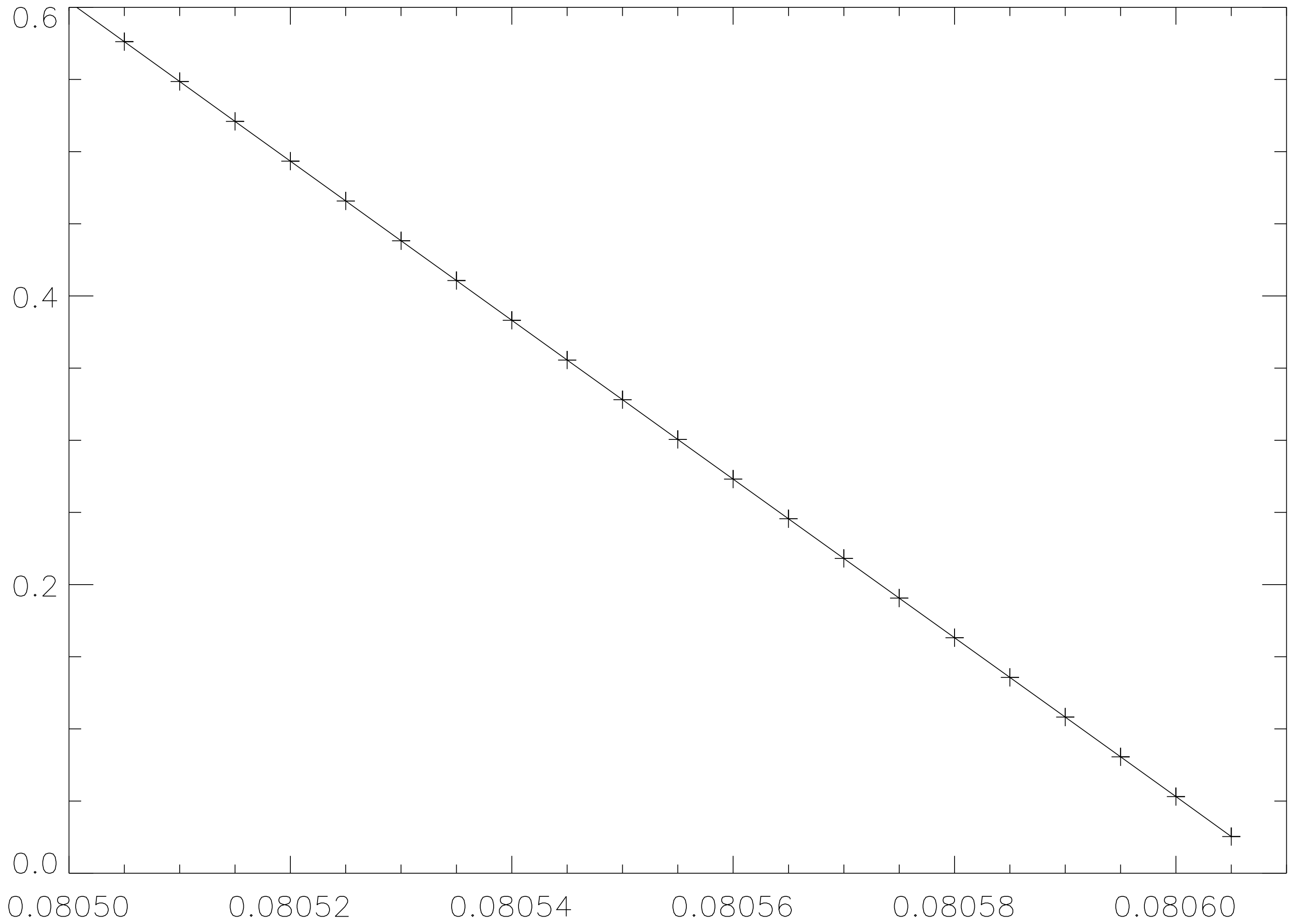


$\min |\Omega|^{-1}$



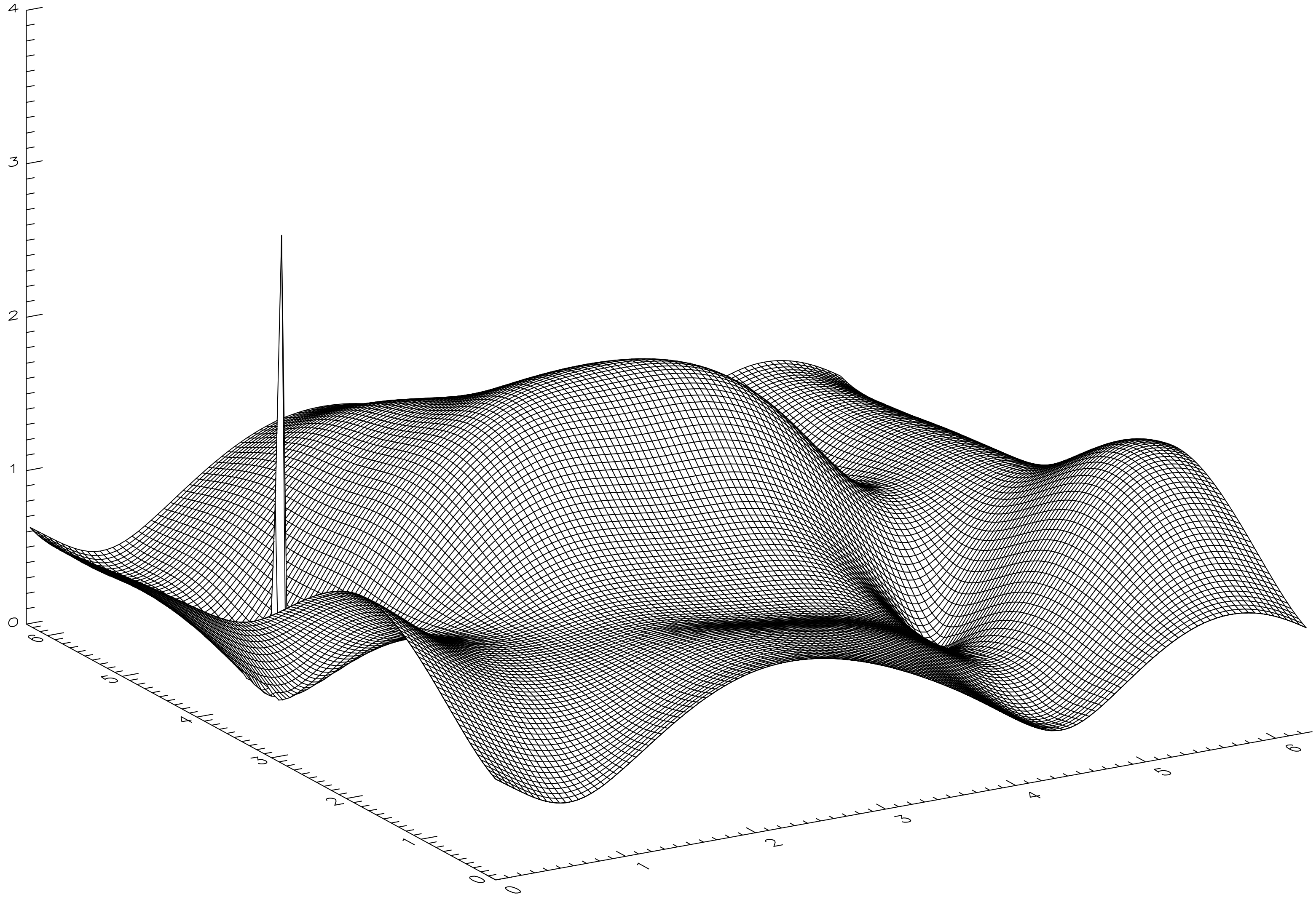
Numerics

$\min J$ (better resolution)



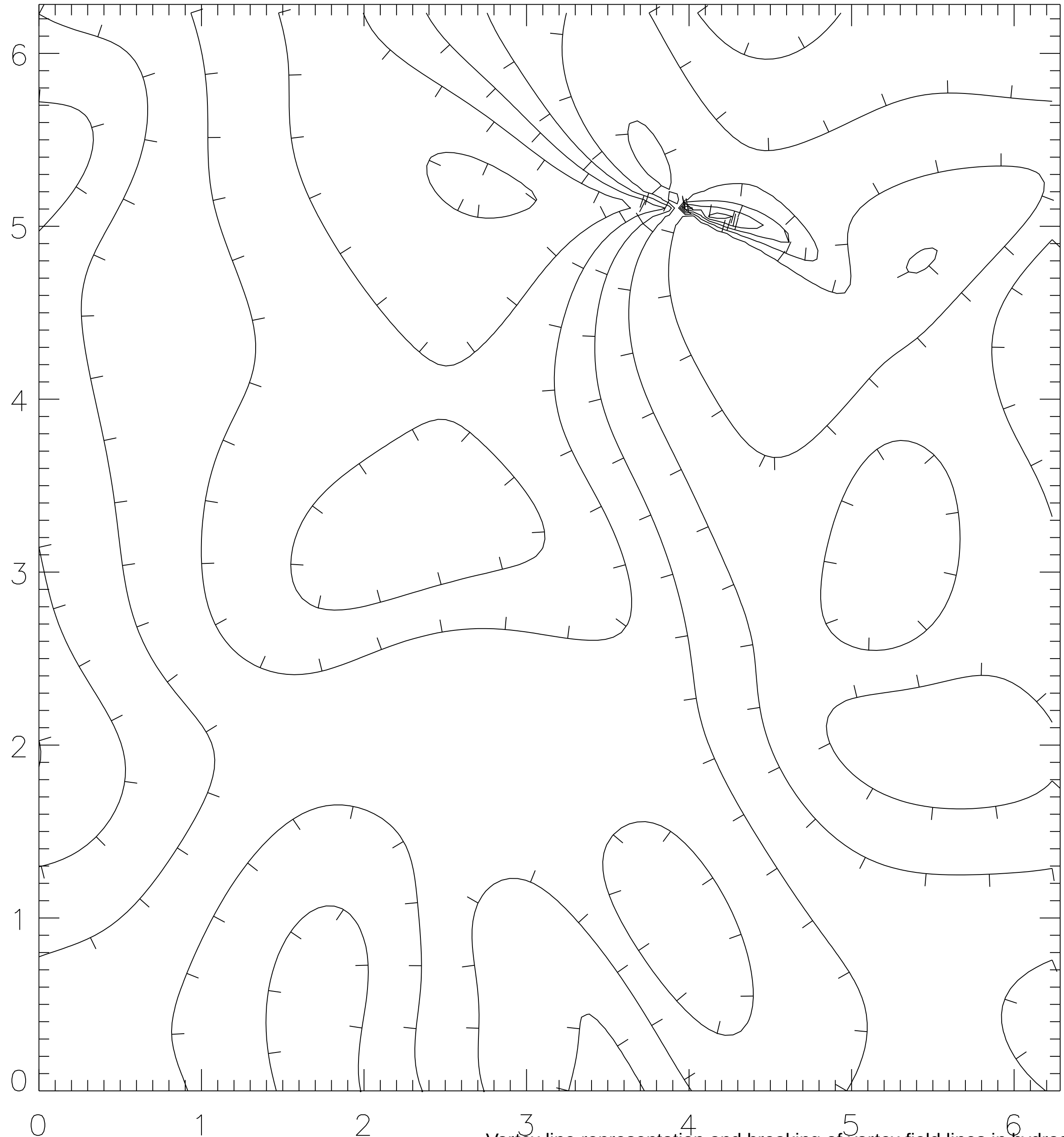
"Random" initial conditions

Dependence of $|\Omega|$ on two coordinates



"Random" initial conditions

Level lines of $|\Omega|$



Numerics

We have checked also that the coefficients γ in the expansion $J(a,t) = \alpha(t_0 - t) + \gamma_{ij}\Delta a_i \Delta a_j$ slightly depend on time as $t \to t_0$.

Concluding remarks

- We have presented the arguments both analytical and numerical in favor of existence of collapse in 3D Euler as the process of breaking of vortex lines.
- The Euler equations under VLR transform into the equations for a compressible charge fluid moving in the self-consistent electromagnetic field.
- Compressible character of VLR is the main reason of breaking of vortex lines. For 3D Euler our numerics show that blowup of the vorticity is connected with vanishing of the Jacobian.

Role of viscosity

We have analysed here only the first stage of breaking described by the Euler equations ($Re \gg 1$).
Near singularity we should use the Navier-Stokes (NS) equations.
The VLR application to the NS SPLITS inertial dynamics:

$$\dot{\mathbf{r}} = \mathbf{v}_n(\mathbf{r}, t),\ \mathbf{r}|_{t=0} = \mathbf{a},$$
$$\mathbf{\Omega}(\mathbf{r}, t) = \text{curl}_r\, \mathbf{v}(\mathbf{r}, t),\ \text{div}_r\, \mathbf{v}(\mathbf{r}, t) = 0$$

$$\mathbf{\Omega}(\mathbf{r}, t) = \frac{(\mathbf{\Omega}_0(\mathbf{a}, t) \cdot \nabla_a)\mathbf{r}(\mathbf{a}, t)}{J(\mathbf{a}, t)},$$

Role of viscosity

and viscous dynamics:

$$\frac{\partial \mathbf{\Omega}_0}{\partial t} = -\nu \ \mathrm{curl}_a \left(\frac{\hat{g}}{J} \, \mathrm{curl}_a \left(\frac{\hat{g}}{J} \, \mathbf{\Omega_0} \right) \right)$$

where ν is the viscosity coefficient and the metric tensor $g_{\alpha\beta} = \frac{\partial x_i}{\partial a_\alpha} \cdot \frac{\partial x_i}{\partial a_\beta}$ (Kuznetsov, 2002). These equations can be used for construction of caustic ($\nu \to 0$).

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