Some Physical Applications of Weak Turbulent Theory

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Kolmogorov-Obukhov spectrum

If the fluid is stirred by external forces at scale of order L, the spectral peak of the excited turbulence is posed at $k \sim k_0 \sim 1/L$. But right behind the peak the spectrum has the universal form:

$$I_k =< |V_k|^2 >\simeq \alpha P^{2/3} k^{-11/3}$$

Here P is the flux of energy to high wave numbers, defined by an intensity of stirring. This is a famous Kolmogorov-Obukhov spectrum. If $P \gg k_0^4 \nu^3$, the Kolmogorov-Obukhov spectrum is realized inside the "interval of universality"

$$k_0 < k < k_m, \qquad k_m \simeq \left(\frac{P}{\nu^3}\right)^{1/4}$$

Kraichnan [1967]: The inverse cascade takes place in $2D$ turbulence of ideal incompressible fluid.

$$K = \frac{1}{2} \int |curlV|^2 d\vec{r}, \qquad curlV = \frac{\partial V_x}{\partial V_y} - \frac{\partial V_y}{\partial V_x}$$

In the area of small scales $k > k_0$ another Kolmogorov-type spectrum is established:

$$I_k \simeq \alpha_q \frac{Q^{2/3}}{k^5}$$

Zakharov [1966]:

Direct Cascade: $$I_k = \alpha_p \frac{P^{1/3}}{k^{7/2}}, \quad \alpha_p = 0,219$$

Inverse Cascade: $$I_k = \alpha_q \frac{Q^{1/3}}{k^{10/3}}, \quad \alpha_q = 0.227$$

Turbulence of capillary waves: direct cascade of energy

Capillary waves on the surface of deep incompressible fluid are described by the following Hamiltonian:

$$H = H_0 + H_1$$

$$H_0 = \int \omega_k a_k^* a_k dk, \quad H_1 = \frac{1}{2}\int V_{kk_1k_2}(a_k^* a_{k_1} a_{k_2} + a_k a_{k_1}^* a_{k_2}^*)\delta_{k-k_1-k_2} dk_1 dk_2 dk_3$$

Here a_k is the complex normal amplitude of capillary ripplons, $\omega_k = (\sigma k^3)^{1/2}$ is the dispersion law of ripplons, σ is coefficient of surface tension, $V_{kk_1k_2} \simeq \sigma^{1/4} k^{9/4}$ is a coupling coefficient of three-ripplon interaction. It is important that $V_{kk_1k_2}$ is a homogenous function:

$$V_{\epsilon k, \epsilon k_1, \epsilon k_2} = \epsilon^{g/4} V_{kk_1k_2}$$

The ripplon's wave action n_k is introduced as follow:

$$< a_k a_{k'}^* >= n_k \delta_{k-k'}$$

Apparently $n_k \dot{k}^2 \gg h$, thus the problem is pure classical. n_k obeys kinetic equation

$$\frac{\partial n_k}{\partial t} = st(n_k), \qquad st(n_k) = \int [R_{kk_1k_2} - R_{k_1kk_2} - Rk_2kk_1] dk_1 dk_2$$

$$R_{kk_1k_2} = 4\pi |V_{kk_1k_2}|^2 \delta_{k-k_1-k_2} \delta_{\omega_k - \omega_{k_1} - \omega_{k_2}} [n_{k_1} n_{k_2} - n_k n_{k_1} - n_k n_{k_2}]$$

Thereafter we assume that spectrum is isotropic, $n = n(|k|)$, and introduce the energy spectrum:

$$\epsilon(\omega) = \omega n(\omega), \qquad n(\omega) = 2\pi k \frac{dk}{d\omega} n(\omega)$$

The isotropic kinetic equation can be written as follow:

$$\frac{\partial \epsilon_\omega}{\partial t} + \frac{\partial P}{\partial \omega} = 0$$

Here $P = -2\pi \int_0^\omega \omega k \frac{dk}{d\omega} st d\omega$ is a flux of energy to high frequency region.

The stationary kinetic equation $st = 0$ is equivalent to $\partial P/\partial\omega = 0, \quad P = P_0$. If $P_0 = 0$, this equation has unique thermodynamic Rayley-Jean's solution:

$$n(\omega) = \frac{T}{\omega}$$

In a general case, it has two-parametric solution:

$$n(\omega) = P^{1/2}\omega^{-17/6} f\left(\frac{T}{P^{1/2}}\omega^{11/6}\right)$$

We must put $T = 0$ (the temperature of turbulence is zero!). Then we get:

$$n(\omega) = \alpha P^{1/3}\omega^{17/6}, \quad \alpha = f(0) \quad \text{is Kolmogorov constant}$$

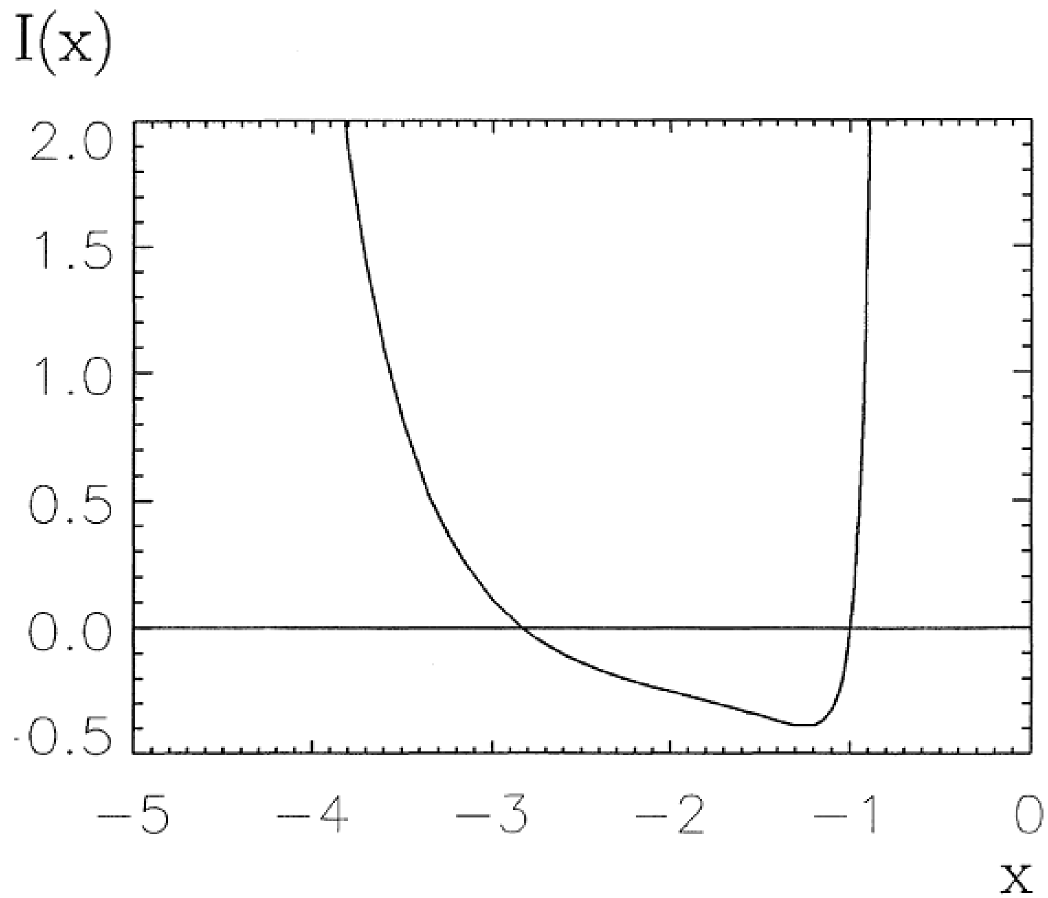
For the surface elevation spectrum $I(\omega) =< |\omega|^2 >$ we get:

$$I(\omega) = \alpha\sigma^{2/3}\omega^{-19/6}$$

Existence of physically significant solution could be rigorously proven both analytically and numerically. For the numerical proof we can apply "frequency variables" $\omega_i = \sqrt{\sigma k_i^3}$ and look for a solution in a powerlike form $n(\omega) = A\omega^x$. Then

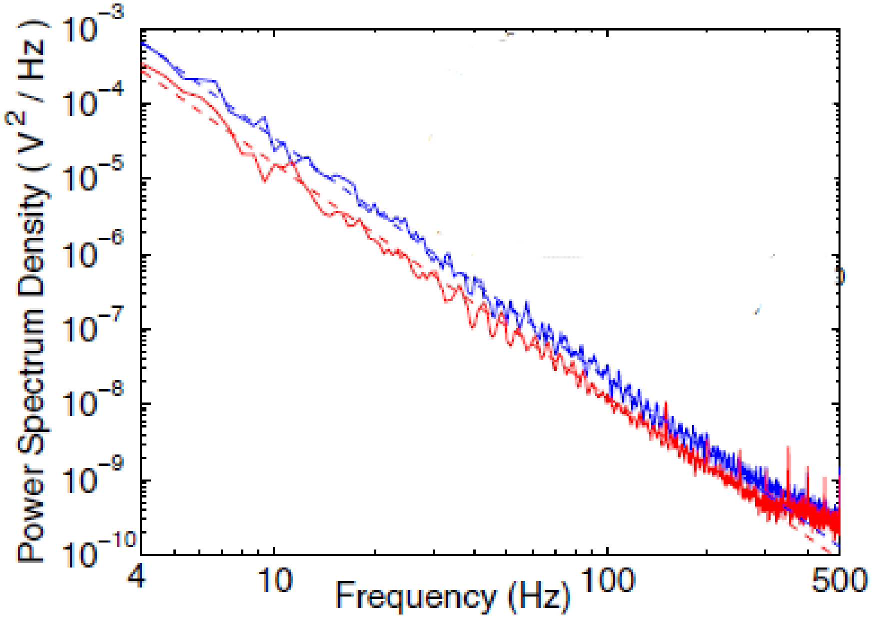
$$st = \frac{A^2 \omega^{-2x+7/6}}{36\pi\sigma^{5/3}} I(x)$$

The integrals in st are convergent if $-5 < x < -5/6$. Function $I(x)$ is presented on next picture.



Collision integral $I(x)$ as a function of solution index x: it is finite within the locality interval $-5 < x < -5/6$ and has two zeros at $x_0 = -1$ (thermodynamical spectrum) and $x = -17/6$ (Kolmogorov solution).

KZ-spectrum for capillary waves was observed in many laboratory experiments. The most advanced one, performed in absence of gravity in free falling plane, showed that law $I(\omega) \simeq \omega^{-19/6}$ was realized with a very good accuracy in the range of two decades.



Power spectrum density of surface wave height *in low gravity*. Lower curve: random forcing 0–6 Hz. Upper curve: sinusoidal forcing at 3 Hz. Dashed lines had slopes of −3.1 (lower) and −3.2 (upper).

Inverse cascade in Bose-condensation

Weakly non-ideal Bose-gas (classical wave approximation, dimensionless variables):

$$i\Psi_t = \Delta\Psi - |\Psi|^2\Psi \quad \text{– Nonlinear Schrodinger equation (NLSE)}$$

Approximation of weak nonlinearity (in the absence of condensate):

$$< \Psi(k)\Psi^*(k') >= n_k \delta_{k-k'} \quad \text{– Particle distribution function for bose-particles}$$

This function obeys kinetic equation:

$$\frac{\partial n}{\partial t} = st(n,n,n) =$$

$$4\pi \int (n_{k_1} n_{k_2} n_{k_3} + n_k n_{k_2} n_{k_3} - n_k n_{k_1} n_{k_2} - n_k n_{k_1} n_{k_3}) \delta_{k+k_1-k_2-k_3} \delta_{k^2+k_1^2-k_2^2-k_3^2} dk_1 dk_2 dk_3$$

The first task of weak turbulent theory - classification of solutions of equation:

$$st(n,n,n)=0$$

We look only for isotropic solutions that depend on $\omega = k^2$. Moreover, we assume $n(\omega) = A\omega^{-x}$. After some transformations we get:

$$st = \pi^3 A^3 \omega^{-3x+2} F(x)$$

$$F(x) = \int_0^1 d\xi \int_1^\infty d\eta [1+(\xi+\eta-1)^x - \xi^x - \eta^x][1+(\xi+\eta-1)^{3x-7/2} - \xi^{3x-7/2} - \eta^{3x-7/2}]$$

$F(x)$ has four solutions: $x=0$, $x=1$, $x=7/6$, $x=3/2$. However, the integral converges only if

$$1/2 < x < 3/2$$

Thus two solutions only make sense:

$$n_1(k) = \frac{T}{k^2} \quad \text{– This is thermodynamic Rayley-Jean spectrum}$$

$$n_2(k) = \frac{\alpha Q^{1/3}}{k^{7/3}} \quad \text{– This is KZ spectrum for inverse cascade}$$

Q is the flux of particles to small scales. Only these two spectra are "local"; two other are "nonlocal" and cannot be realized. Kinetic equation for particle distribution function has a family of self-similar solutions depending on arbitrary parameter β:

$$n \simeq \frac{1}{|t_0 - t|^{2\beta+1/2}} F\left(\frac{k}{(t_0 - t)^{\beta}}\right)$$

In this case $Q \simeq (t_0 - t)^{\beta - 3/2}$. Compensation takes place if $\beta > 0,\ k_n \simeq (t_0 - t)^{\beta}$. To define β we must know the rate of cooling of Bose-gas.

Theory of wind-driven sea: coexistence of direct and inverse cascades

If $\omega_k = \sqrt{gk}$ is dispersion relation of gravity waves on deep water, the lowest-order nonlinear process in the wind-driven sea is four-wave interaction governed by resonant condition:

$$\vec{k_1} + \vec{k_2} = \vec{k_3} + \vec{k_4}, \quad \omega_{k_1} + \omega_{k_2} = \omega_{k_3} + \omega_{k_4}$$

Gravity surface waves are described by Hamiltonian:

$$\int \omega_k b_k^* b_k dk + \frac{1}{4} \int T_{kk_1,k_2k_3} b_k^* b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3$$

b_k is "renormalized" complex normal amplitude of gravitational ripplons. $T_{kk_1k_2k_3}$, the coupling coefficient of four-wave interaction, is a complicated homogenous function of third order:

$$T_{\epsilon k,\epsilon k_1,\epsilon k_2,\epsilon k_3} = \epsilon^3 T_{kk_1k_2k_3}$$

We introduce the wave action spectrum of ripplons by standard way:

$$< b_k b^*_{k'} >= n_k \delta_{k-k'}$$

A slightly inhomogeneous wind-driven sea n_k obeys the Hasselmann kinetic equation:

$$\frac{\partial n_k}{\partial t} = S_{nl} + S_{in} + S_{diss}$$

$$S_{nl} = F_k - n_k \Gamma_k$$

$$F_k = \pi \int |T_{kk_1k_2k_3}|^2 n_{k_1} n_{k_2} n_{k_3} \delta_{k+k_1-k_2-k_3} b_{\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3}} dk_1 dk_2 dk_3$$

$$\Gamma_k = \pi \int |T_{kk_1k_2k_3}|^2 (n) k_1 n_{k_2} + n_{k_1} n_{k_3} - n_{k_2} n_{k_3}) \delta_{k+k_1-k_2-k_3} \delta_{\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3}} dk_1 dk_2 dk_3$$

Γ_k can be treated as an "effective dissipation" due to nonlinear four-wave interaction.

The term S_{in} appears due to interaction with wind.

The term $S_{diss} < 0$ is the rate of dissipation.

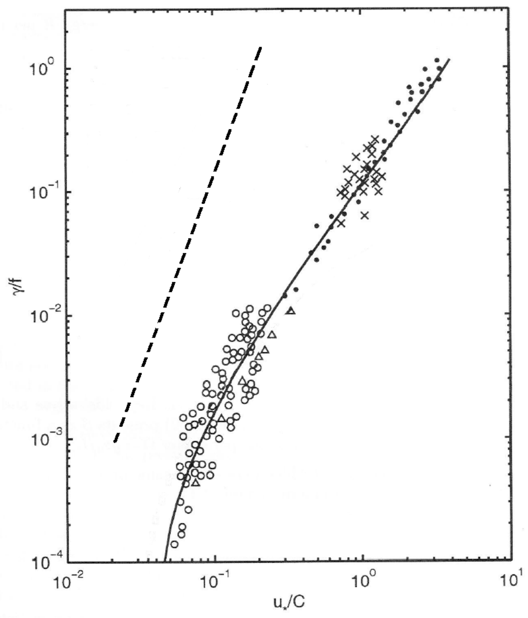
$$S_{in} + S_{diss} > 0$$

$$S_{in} = \gamma_k n_k, \quad \gamma_k \quad \text{is the growth rate of instability}$$

Γ_k surpasses γ_k at least in order of magnitude!

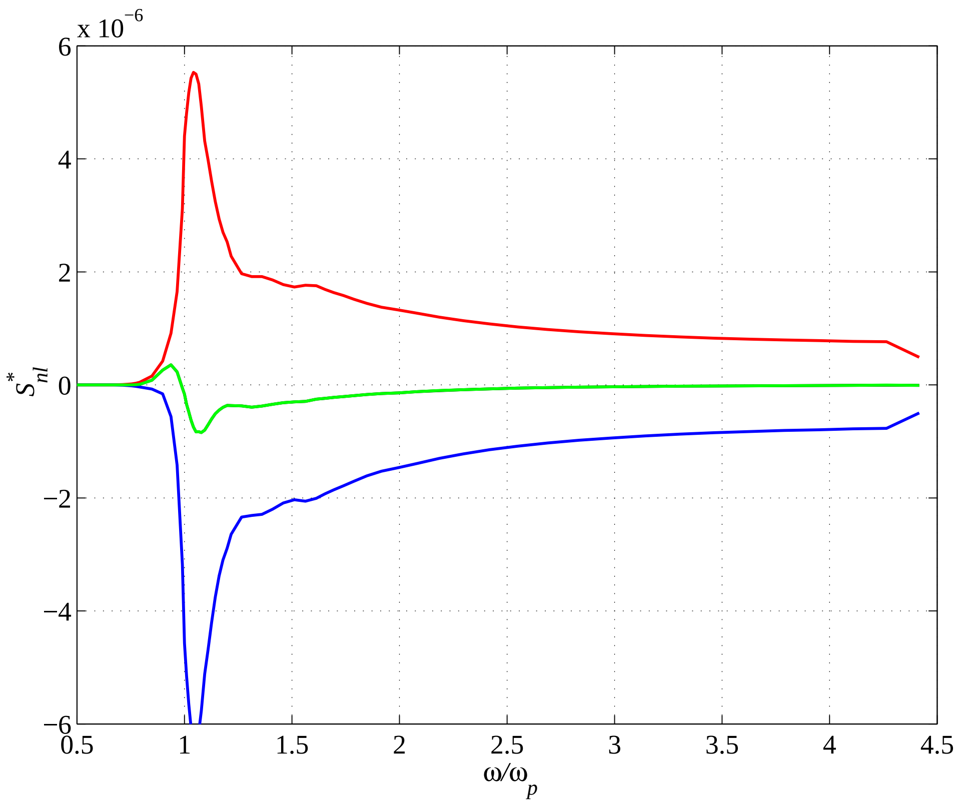
The "conservative" equation:

$$\frac{\partial n_k}{\partial t} = S_{nl} = F_k - n_k \Gamma_k$$



Dots and solid line: experimental data for the wind-induced growth rate $2\pi\,\gamma_{in}(\omega)/\omega$. Dashed line: the damping due to four-wave interactions $2\pi\,\Gamma(\omega)/\omega$, calculated for narrow in angle spectrum at $\mu \simeq 0.05$ using equation:

$$\Gamma_\omega = 36\,\pi\omega \left(\frac{\omega}{\omega_p}\right)^3 \mu_p^4 \cos^2\theta$$



Split of nonlinear interaction term S_{nl} (central curve) into F_k (upper curve) and $\Gamma_k\, N_k$ (lower curve). For nonlinear interaction term $S_{nl} = F_k - \Gamma_k\, N_k$ the magnitudes of constituents F_k and $\Gamma_k\, N_k$ essentially exceed their difference. They are one order higher than the magnitude of S_{nl}!

The "conservative" equation

$$\frac{\partial n_k}{\partial t} = S_{nl} = F_k - n_k \Gamma_k$$

conserves two motion constants, energy and momentum:

$$E = \int \omega_k n_k dk \qquad \vec{P} = \int \vec{k} n_k dk$$

and the total wave action (or number of particles):

$$N = \int n_k d\vec{k}$$

In the isotropic situation $\vec{P} = 0$.

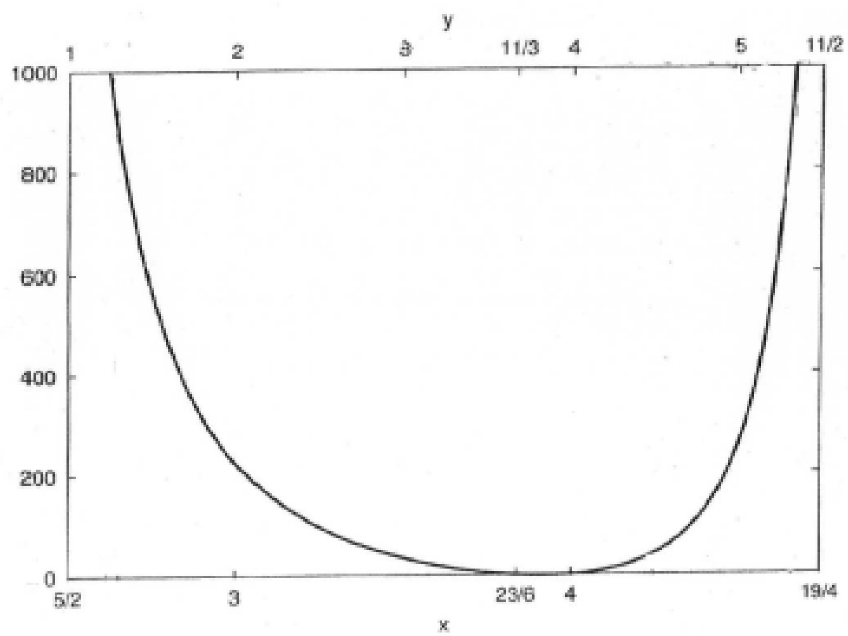
The central question of theory is classification of solutions of this equation:

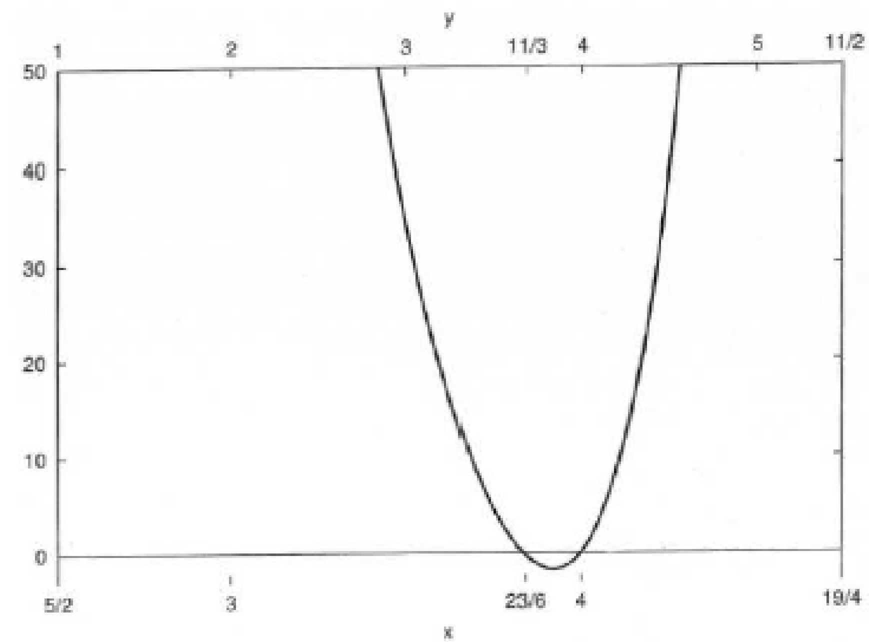
$$S_{nl} = 0$$

The isotropic powerlike solutions $N_k = k^{-x}$.

$$S_{nl} = g^{3/2} k^{-3x+19/2} F(x)$$

The integral in S_{nl} converges if $3/2 < x < 19/4$. $F(x) = 0$ if $F(x)$ has only two zeros: $x = 4$ and $x = 23/3$. The numerical experiment confirms this fact with a great accuracy.





(a): Plot of function $F(x)$.

(b): Plot of function $F(x)$. Zoom in the vertical direction.

$$N_k^{(1)} = c_p \left(\frac{P_0}{g}\right)^{1/3} \frac{1}{k^4}$$

$$N_k^{(2)} = c_q \left(\frac{Q_0}{q^{3/2}}\right)^{1/3} \frac{1}{k^{23/6}}$$

In a typical situation the spectrum has a peak at $k \sim k_0$ and the "rear face" at $k > k_0$. This is a field where inverse and direct cascades coexist and compete.

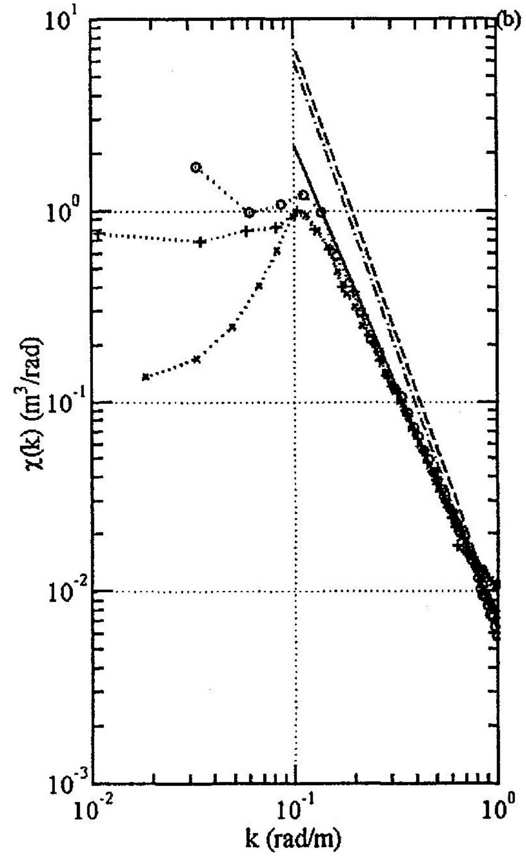
The angle energy spectrum:

$$\epsilon_k \simeq 2\pi k \omega_k N$$

For direct cascade: $\epsilon_k \simeq k^{-5/2} \simeq k^{-2.5}$

For inverse cascade: $\epsilon_k \simeq k^{-7/3} \simeq k^{-2.33}$

In reality: $\epsilon_k \simeq k^{-x}$, where $2.33 < x < 2.5$



Surface elevation spectra proportional to $E(k)/g$ with comparison slopes. Solid line: $k^{-5/2}$; dashed and dashed dotted curves: k^{-3} (Phillips's spectrum) with different normalizations.

Waves forecasting.