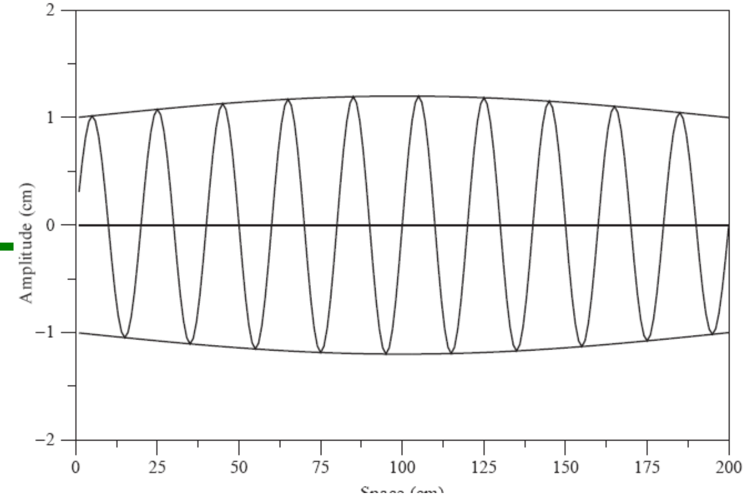
Nonlinear stage of modulation instability

A.A. Gelash V.E. Zakharov

1 Uniformization

NLSE $$i\varphi_t - \tfrac{1}{2}\varphi_{xx} - (|\varphi|^2 - A^2)\varphi = 0$$

NLSE describes the envelope

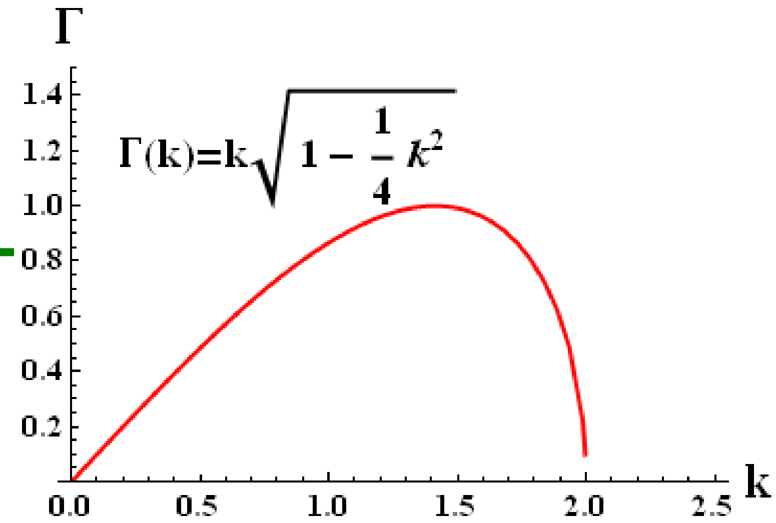


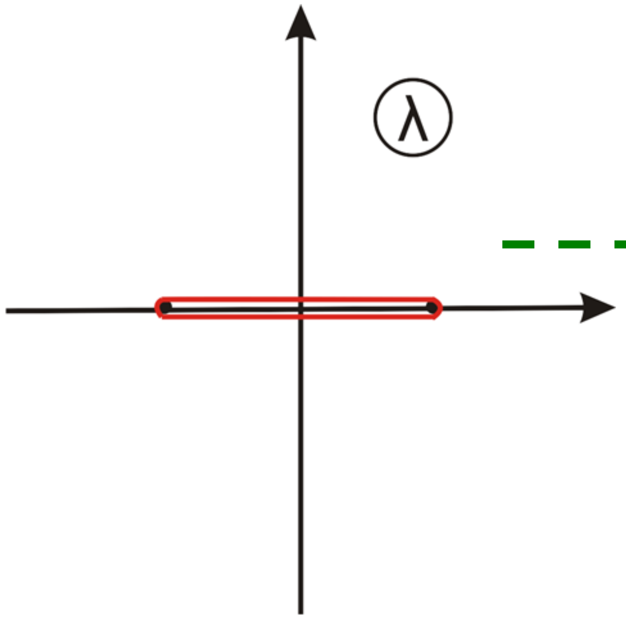
We study NLSE with nonvanishing boundary conditions.
Instead of NVBC we use the term "in presence of condensate"

$$|\varphi|^2 \to |A|^2 \quad at \quad x \to \pm\infty$$

Uniformization

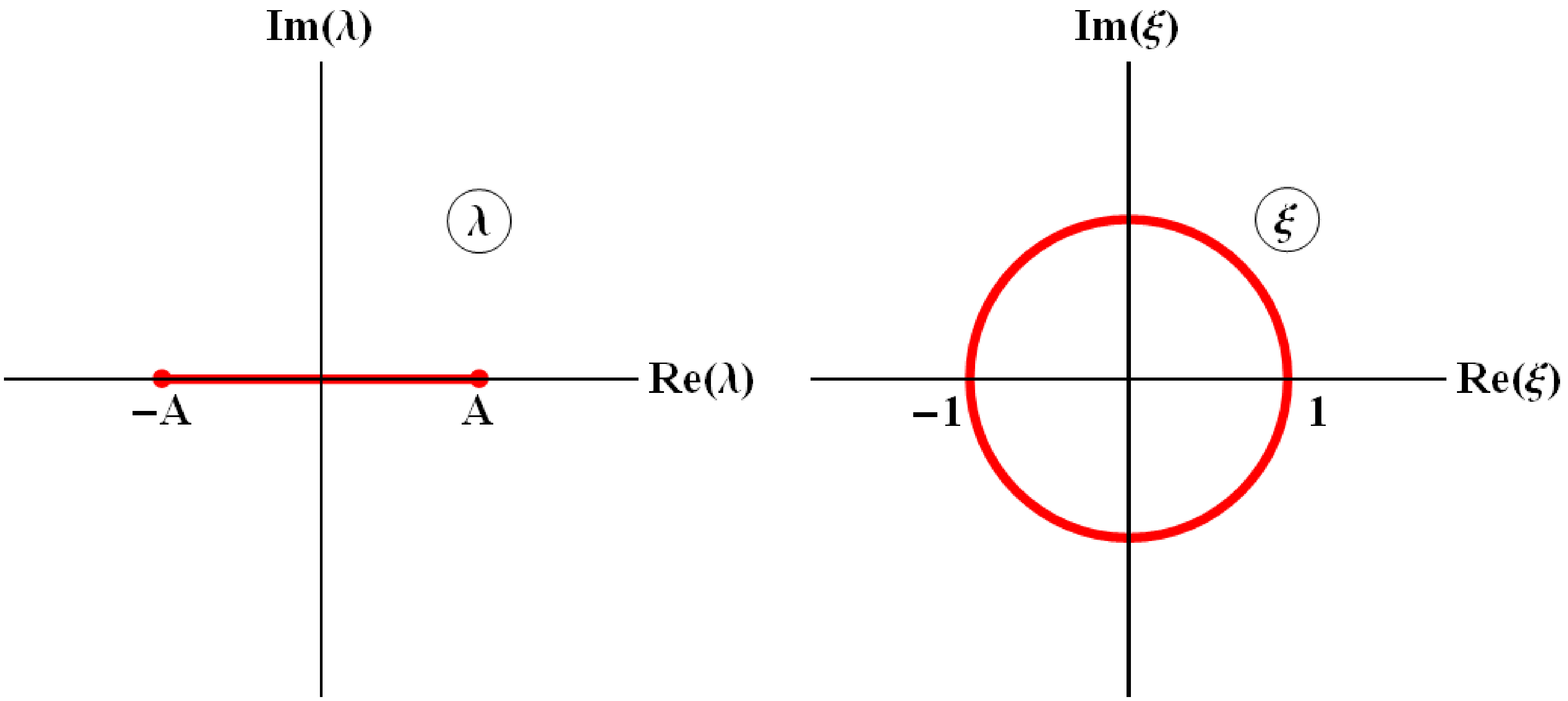
The increment of modulation instability





The plane of spectral parameter

3 Uniformization



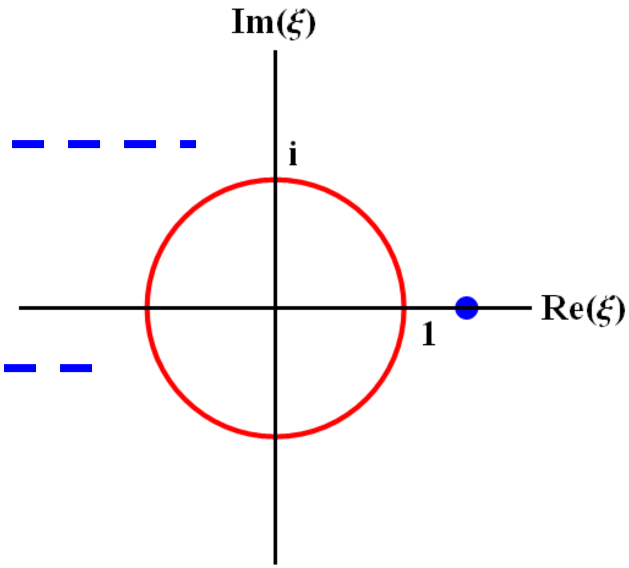
Jukowsky map :

$$\lambda = \frac{A}{2}\left(\xi + \frac{1}{\xi}\right) \qquad k = \frac{A}{2}\left(\xi - \frac{1}{\xi}\right) \qquad \xi + \xi^{*} \neq 0$$

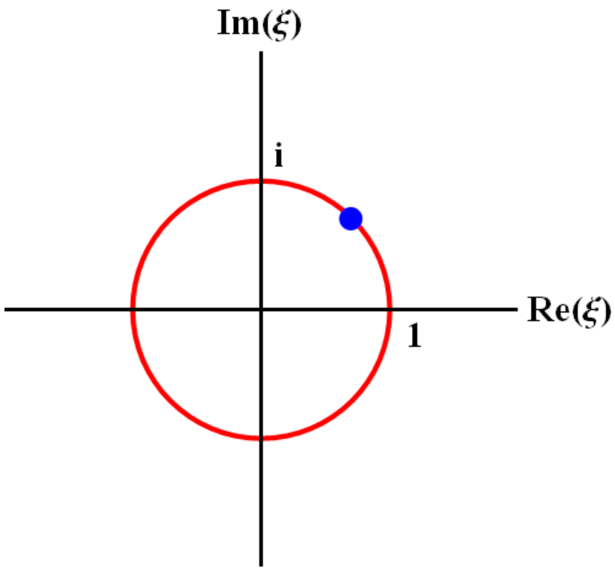
4 Previous works. NLSE with presence of condensate

Evgenii A. Kuznetsov (1977)

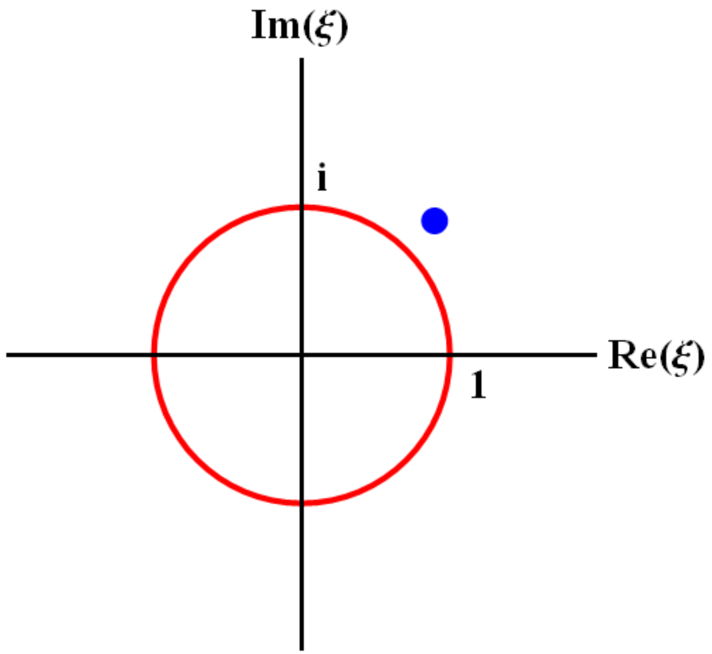
Yan-Chow Ma (1979)



N. N. Akhmediev and V.I.Korneev (1979)



Previous works. General one-solitonic solution



A.R. Its, A.V. Rybin and M.A. Sall (1988)
M. Tajiri and Y. Watanabe (1998)
Q.-Han Park and H. J. Shin (1999)
A. Slunyaev, C. Kharif, E. Pelinovsky, T. Talipova (2002)
S. L. Lu Li, Zhonghao Li and G. Zhou (2004).
N. Akhmediev, J. M. Soto-Crespo, A. Ankiewicz (2009)
V. E. Zakharov and A. A. Gelash, (2011)
…

NLSE via dressing method

$$i\varphi_t - \frac{1}{2}\varphi_{xx} - (|\varphi|^2 - |A|^2)\varphi = 0$$

is the compatibility condition for the following overdetermined linear system

$$\frac{\partial \Psi}{\partial x} = \widehat{U}\Psi$$

$$i\frac{\partial \Psi}{\partial t} = (\lambda \widehat{U} + \widehat{W})\Psi$$

$$\widehat{U} = I\lambda + u, \qquad \widehat{W} = \frac{1}{2}\begin{pmatrix} |\varphi|^2 - A^2 & \varphi_x \\ \varphi_x^* & -|\varphi|^2 + A^2 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad u = \begin{pmatrix} 0 & \varphi \\ -\varphi^* & 0 \end{pmatrix}$$

NLSE via dressing method

Suppose we know some solution φ_0 of the NLSE together with a fundamental solution Ψ_0

$$\frac{\partial \Psi_0}{\partial x} = \widehat{U}_0 \Psi_0$$

$$i\frac{\partial \Psi_0}{\partial t} = (\lambda \widehat{U}_0 + \widehat{W}_0)\Psi_0$$

Then we introduce "the dressing function"

$$\chi = \Psi \Psi_0^{-1}$$

We demand that χ is regular at infinity

$$\chi(\lambda) \to E + \frac{\widetilde{\chi}}{\lambda} + \cdots \qquad \text{at } |\lambda| \to \infty$$

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\varphi = \varphi_0 - 2\widetilde{\chi}_{(12)}$$

NLSE via dressing method

$$\chi_{\alpha\beta} = \delta_{\alpha\beta} + \sum_n \frac{p_{n,\alpha} q_{n,\beta}}{\lambda - \lambda_n}$$

$$q^*_{n\alpha} = \Psi_{0,\alpha\beta}(-\lambda^*_n)\xi_{n\beta}$$

$$\xi_n = \begin{pmatrix} 1 \\ C_n \end{pmatrix} \qquad F_{n\alpha\beta} = \Psi_{0,\alpha\beta}(-\lambda^*_n)$$

$$\sum_m \frac{(\vec{q}_n \cdot \vec{q}^*_m)}{\lambda_n + \lambda^*_m} \vec{p}^*_m = \vec{q}_n$$

N-solitonic solution on condensate

Since this moment we study dressing only on the condensate background

Now one can put $\varphi_0 = A$

$$U_0 = \begin{pmatrix} \lambda & A \\ -A & -\lambda \end{pmatrix} \qquad \widehat{W}_0 = 0$$

And Ψ_0 can be found as

$$\Psi_0(x,t,\lambda) = \frac{1}{\sqrt{1-s^2(\lambda)}} \begin{pmatrix} e^{\phi(x,t,\lambda)} & s(\lambda)\cdot e^{-\phi(x,t,\lambda)} \\ s(\lambda)\cdot e^{\phi(x,t,\lambda)} & e^{-\phi(x,t,\lambda)} \end{pmatrix}$$

Here $\quad \phi = kx + \Omega t, \;\; k^2 = \lambda^2 - A^2, \;\; \Omega = -i\lambda k, \;\; s = -\dfrac{A}{\lambda + k}$

$$\Psi_0^{-1}(x,t,\lambda) = \frac{1}{\sqrt{1-s^2(\lambda)}} \begin{pmatrix} e^{-\phi(x,t,\lambda)} & -s(\lambda)\cdot e^{-\phi(x,t,\lambda)} \\ -s(\lambda)\cdot e^{\phi(x,t,\lambda)} & e^{\phi(x,t,\lambda)} \end{pmatrix}$$

Notice that $\quad k^*(-\lambda^*) = -k(\lambda), \quad s^*(-\lambda^*) = -s(\lambda), \quad \phi^*(-\lambda^*) = -\phi(\lambda)$

Thereafter we denote for simplicity.

$$\phi_n = \phi_n(\lambda_n) \qquad s_n = s(\lambda_n)$$

N-solitonic solution on condensate

One can check that

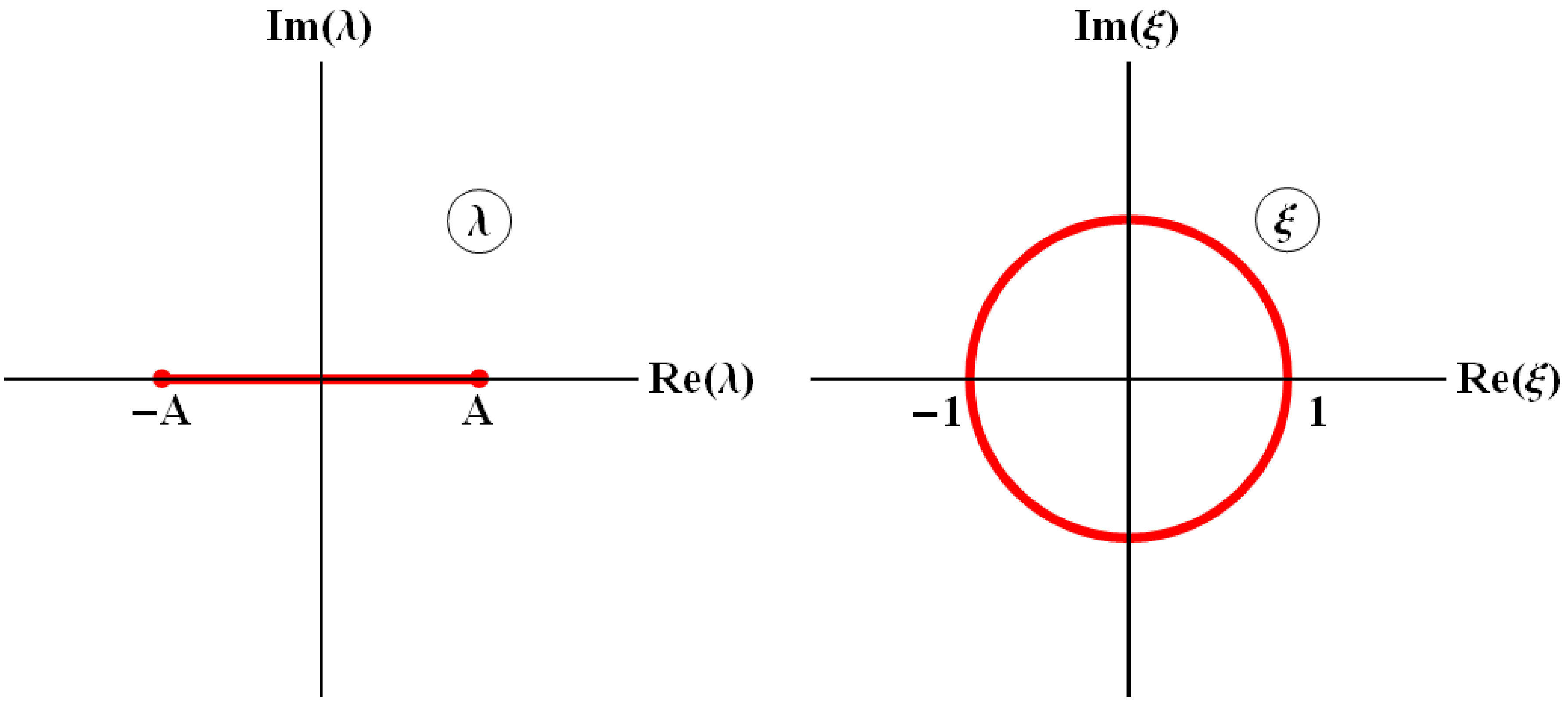
$$\Psi_0^{-1}(-\lambda^*) = \Psi_0^+(\lambda)$$

$$\phi_n(-\lambda_n^*) = -\phi_n^* \qquad s_n(-\lambda_n^*) = -s_n^*$$

$$F_n = \Psi_0(-\lambda_n^*) = \begin{pmatrix} e^{-\phi_n^*} & -s_n^* \cdot e^{\phi_n^*} \\ -s_n^* \cdot e^{-\phi_n^*} & e^{\phi_n^*} \end{pmatrix} \qquad q_n^* = F_n \begin{pmatrix} 1 \\ C_n \end{pmatrix}$$

$$q_{n1} = e^{-\phi_n} - C_n^* s_n e^{\phi_n} \qquad q_{n2} = -s_n e^{-\phi_n} + C_n^* e^{\phi_n}$$

11 Uniformization



Jukowsky map :

$$\lambda = \frac{A}{2}\left(\xi + \frac{1}{\xi}\right) \qquad k = \frac{A}{2}\left(\xi - \frac{1}{\xi}\right) \qquad \xi + \xi^* \neq 0$$

N-solitonic solution on condensate (uniformization)

$$\lambda = \frac{A}{2}(\xi + \frac{1}{\xi}) \qquad k = \frac{A}{2}(\xi - \frac{1}{\xi}) \qquad s = -\frac{1}{\xi} \qquad \xi + \xi^* \neq 0$$

$$\xi_n = R_n e^{i\alpha_n} \qquad C_n = e^{i\theta_n + \mu_n} \qquad R_n = e^{z_n}$$

$$\lambda_n = \frac{A}{2}\left(R_n + \frac{1}{R_n}\right)\cos(\alpha_n) + \frac{iA}{2}\left(R_n - \frac{1}{R_n}\right)\sin(\alpha_n) =$$

$$A\Big[\cosh(z_n)\cos(\alpha_n) + i\sinh(z_n)\sin(\alpha_n)\Big]$$

N-solitonic solution on condensate (uniformization)

$$q_{n1} = e^{-\phi_n} + e^{w_n+\phi_n} \qquad q_{n2} = e^{w_n-\phi_n} + e^{\phi_n}$$

$$\phi_n = u_n + iv_n$$

$$u_n = æ_n x - \gamma_n t + \frac{1}{2}\mu_n \qquad v_n = k_n x - \omega_n t + \frac{1}{2}\theta_n$$

$$æ_n = \frac{A}{2}\left(R_n - \frac{1}{R_n}\right)\cos(\alpha_n) = A\sinh(z_n)\cos(\alpha_n)$$

$$k_n = \frac{A}{2}\left(R_n + \frac{1}{R_n}\right)\sin(\alpha_n) = A\cosh(z_n)\sin(\alpha_n)$$

$$\gamma_n = -\frac{A^2}{4}\left(R_n^2 + \frac{1}{R_n^2}\right)\sin(2\alpha_n) = -\frac{A^2}{2}\cosh(2z_n)\sin(2\alpha_n)$$

$$\omega_n = \frac{A^2}{4}\left(R_n^2 - \frac{1}{R_n^2}\right)\cos(2\alpha_n) = \frac{A^2}{2}\sinh(2z_n)\cos(2\alpha_n)$$

General one-solitonic solution

$$\varphi = -\frac{A}{\cosh(z)\cosh(2u) + \cos(\alpha)\cos(2v)} \times$$

$$\Big[\cosh(z)\cos(2\alpha)\cosh(2u) + \cosh(2z)\cos(\alpha)\cos(2v)$$

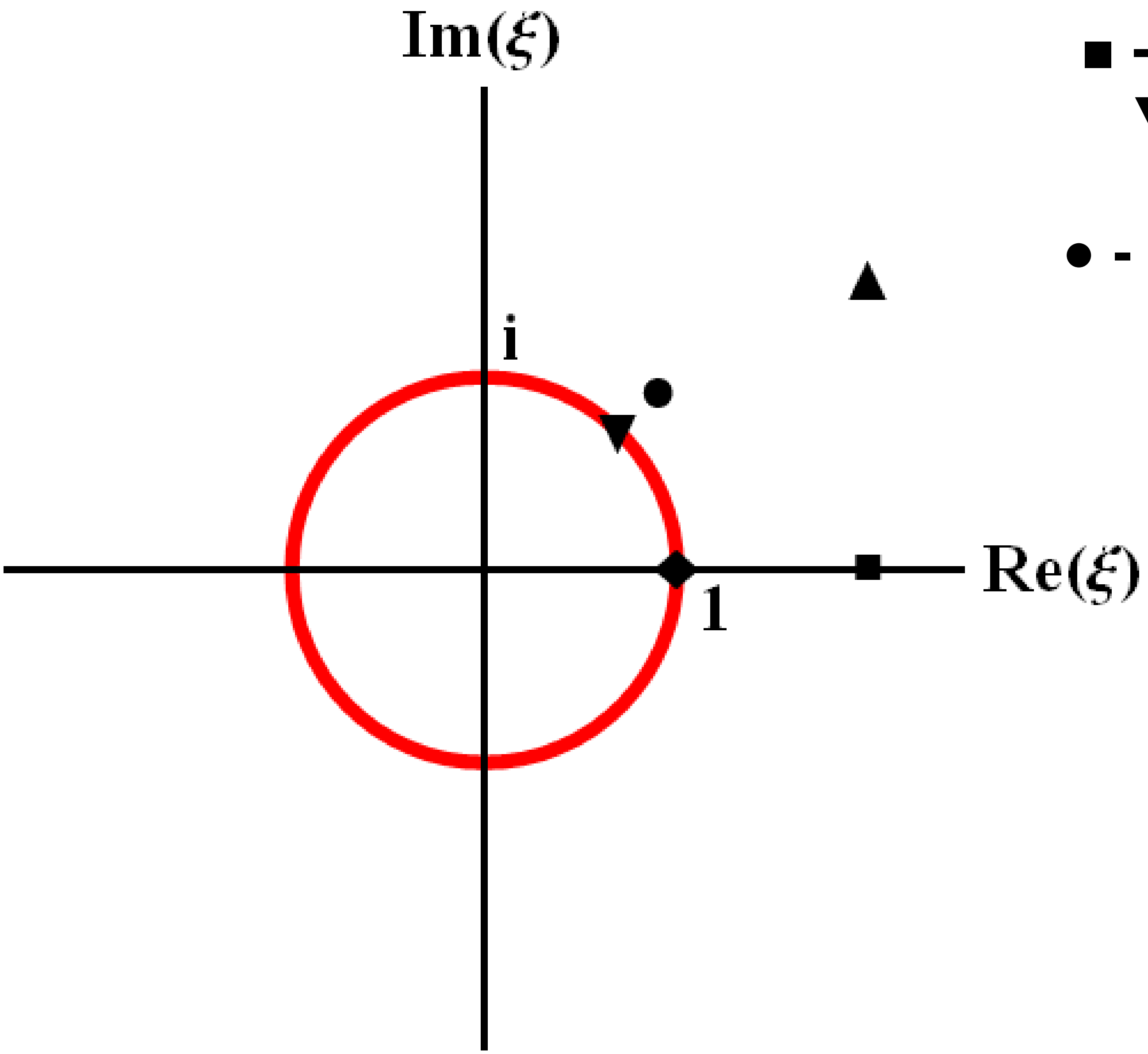
$$+i\Big(\cosh(z)\sin(2\alpha)\sinh(2u) + \sinh(2z)\cos(\alpha)\sin(2v)\Big)\Big]$$

$$u = æx - \gamma t \qquad v = kx - \omega t$$

$$æ = A\sinh(z)\cos(\alpha), \quad \gamma = -\frac{A^2}{2}\cosh(2z)\sin(2\alpha)$$

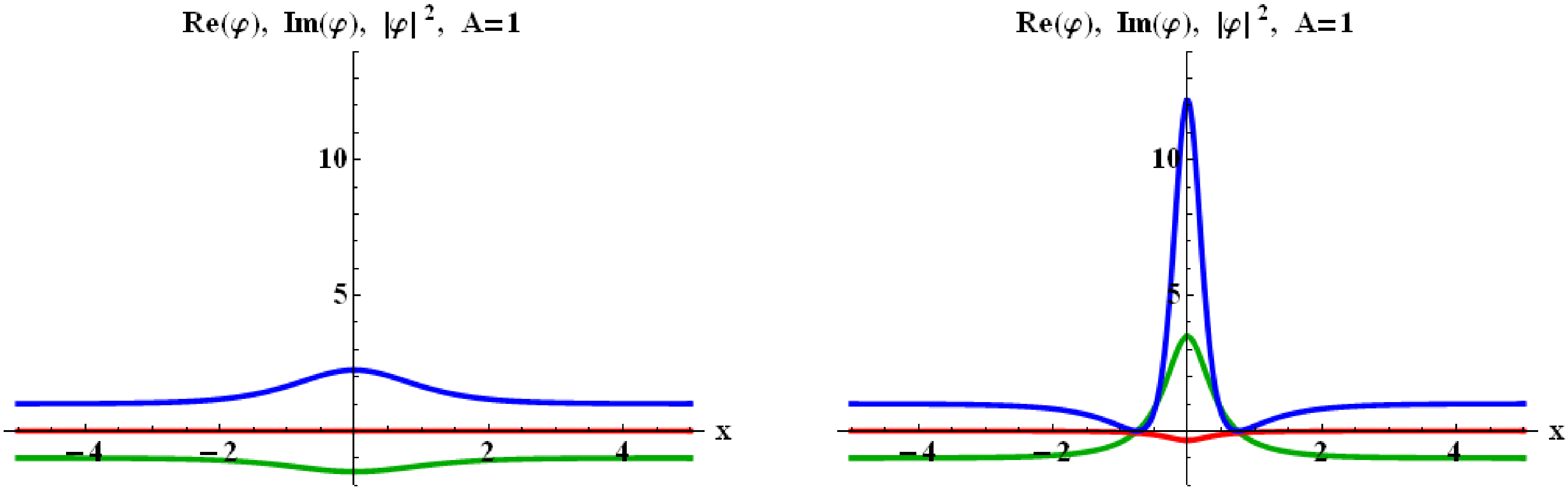
$$k = A\cosh(z)\sin(\alpha), \quad \omega = \frac{A^2}{2}\sinh(2z)\cos(2\alpha)$$

“Species” of one-solitonic solution



■ - Kuznetsov-Ma
▼ - Akhmediev
▲ - Genaral
● - near Akhmediev
♦ - Peregrine

Kuznetsov-Ma soliton



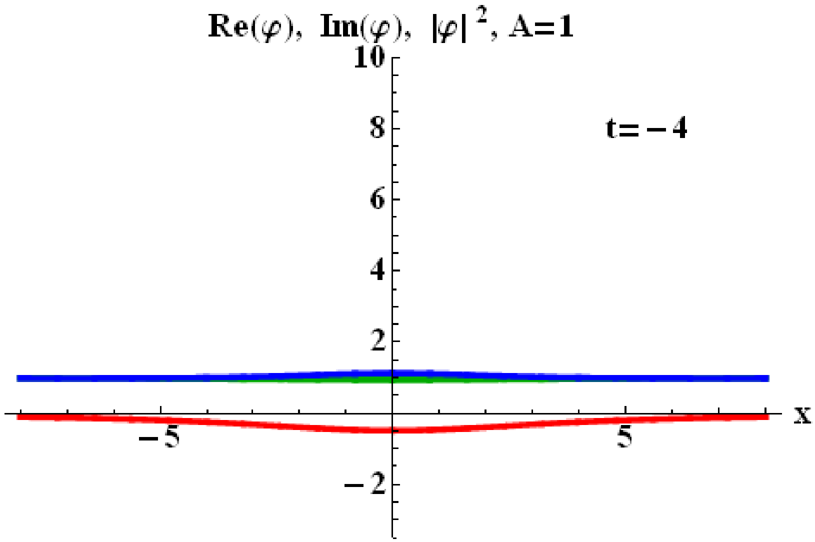
Kuznetsov-Ma soliton at the moment of minimum (left) and maximum (right) of its amplitude. Re – green, Im – Red, Abs^2 - Blue

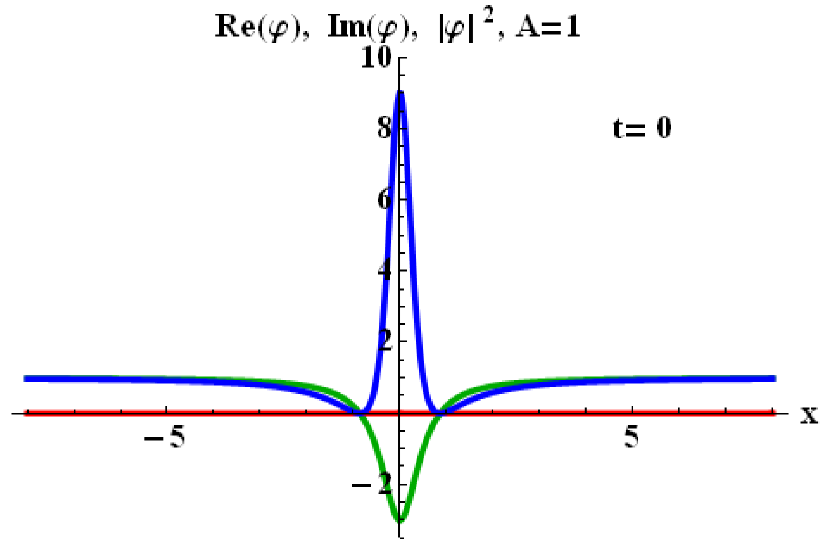
$$\varphi = -\frac{A}{\cosh(z)\cosh(2u) + \cos(2v)}$$

$$\left[\cosh(z)\cosh(2u) + \cosh(2z)\cos(2v) + i\sinh(2z)\sin(2v)\right]$$

$$u = A\sinh(z)x \qquad v = \frac{A^2}{2}\sinh(2z)t$$

Peregrine breather

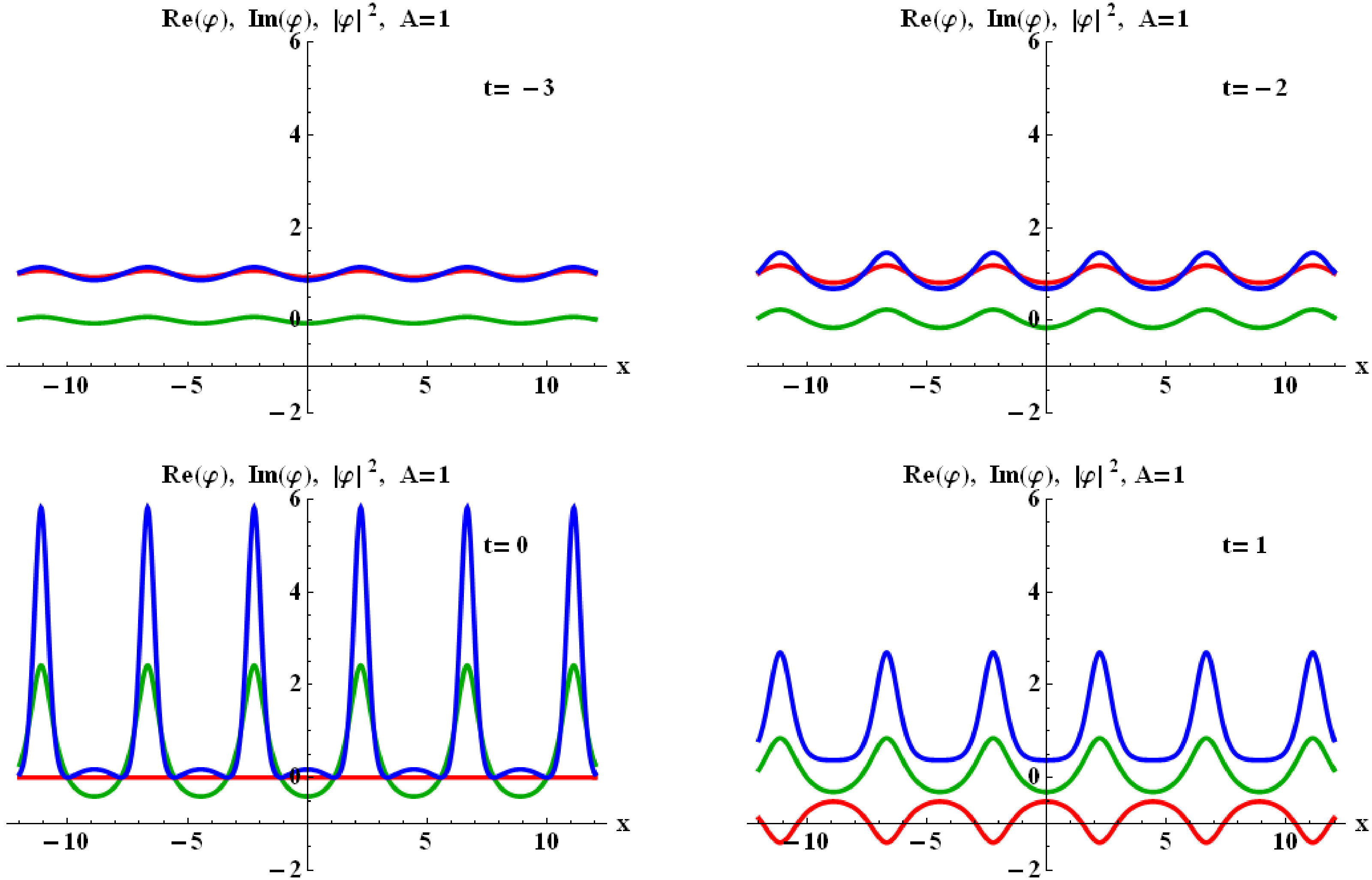




$R = 1, \alpha = 0$ **Re – green, Im – Red, Abs^2 - Blue**

$$\varphi = A\left(1 - 4\frac{1 - 2it}{1 + 4x^2 + 4t^2}\right)$$

Akhmediev breather



$R=1, \alpha = \frac{\pi}{4}$ **Akhmediev breather at different moments of time**

Re – green, Im – Red, Abs^2 - Blue

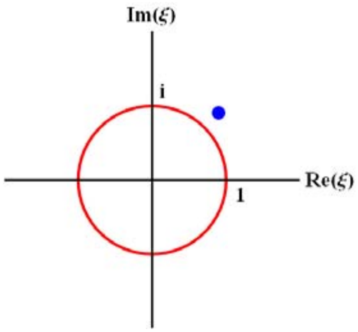
Akhmediev breather

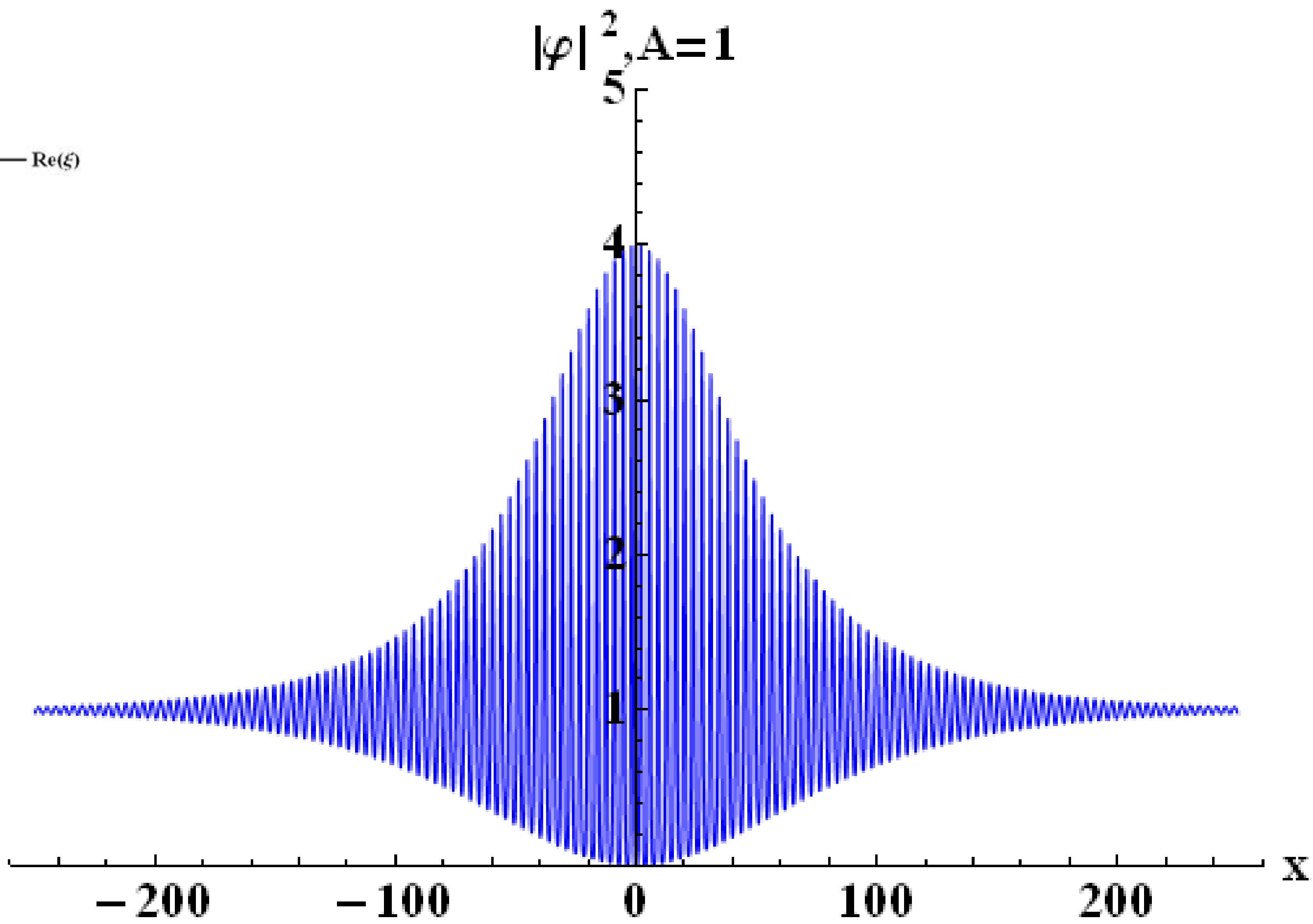
$$\varphi = -A\frac{\cos(2\alpha)\cosh(2u) + \cos(\alpha)\cos(2v) + i\sin(2\alpha)\sinh(2u)}{\cosh(2u) + \cos(\alpha)\cos(2v)}$$

$$u = \frac{1}{2}A^2\sin(2\alpha)t \qquad v = A\sin(\alpha)x$$

$$\varphi \to -A\exp[\mp 2i\alpha] \quad at \quad t \to \mp\infty$$

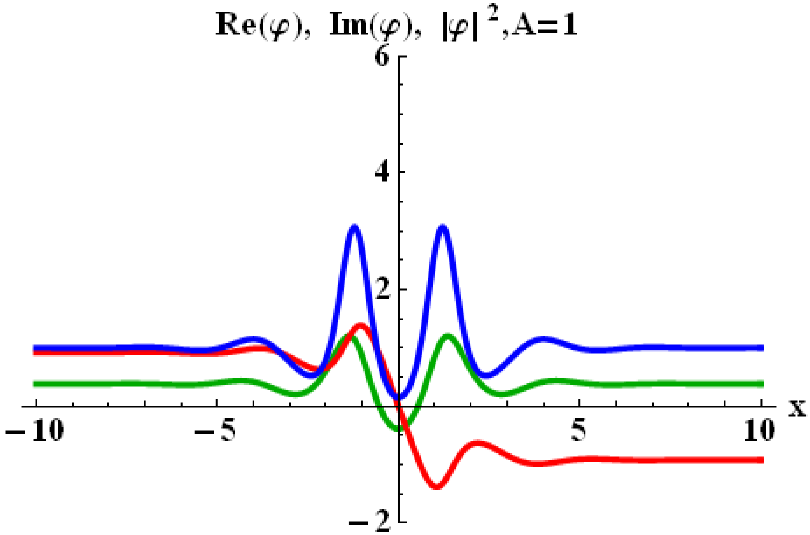
Near-Akhmediev solution

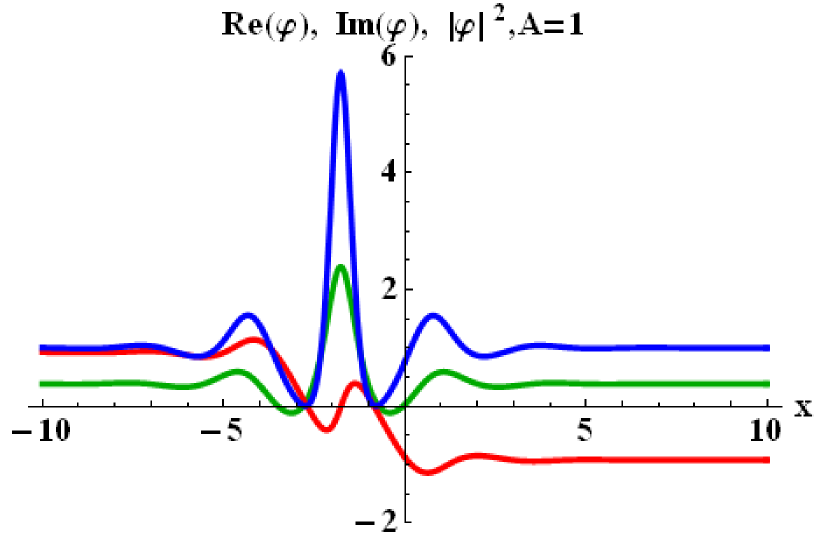




$R = 1.02,\ \alpha = \dfrac{\pi}{3}$ **The "Near-Akhmediev" solution**

General solution





$R = 2, \alpha = \frac{5\pi}{16}$

General solution at the moments of minimum (left) and maximum (right) of its amplitude

$$V_{gr} = \frac{\gamma}{æ} = A\frac{\cosh(2z)}{\sinh(z)}\sin(\alpha) \qquad V_{ph} = 2A\sinh(z)\frac{\cos(2\alpha)}{\sin(\alpha)}$$

Time averaging

$$< |\varphi|^2 >_T = \frac{A^2}{2\pi} \int_0^{2\pi} \frac{1}{\Big(\cosh(z)\cosh(2u) + \cos(\alpha)\cos(\tau)\Big)^2} \times$$

$$\Big[\Big(\cosh(z)\cos(2\alpha)\cosh(2u) + \cosh(2z)\cos(\alpha)\cos(\tau)\Big)^2$$

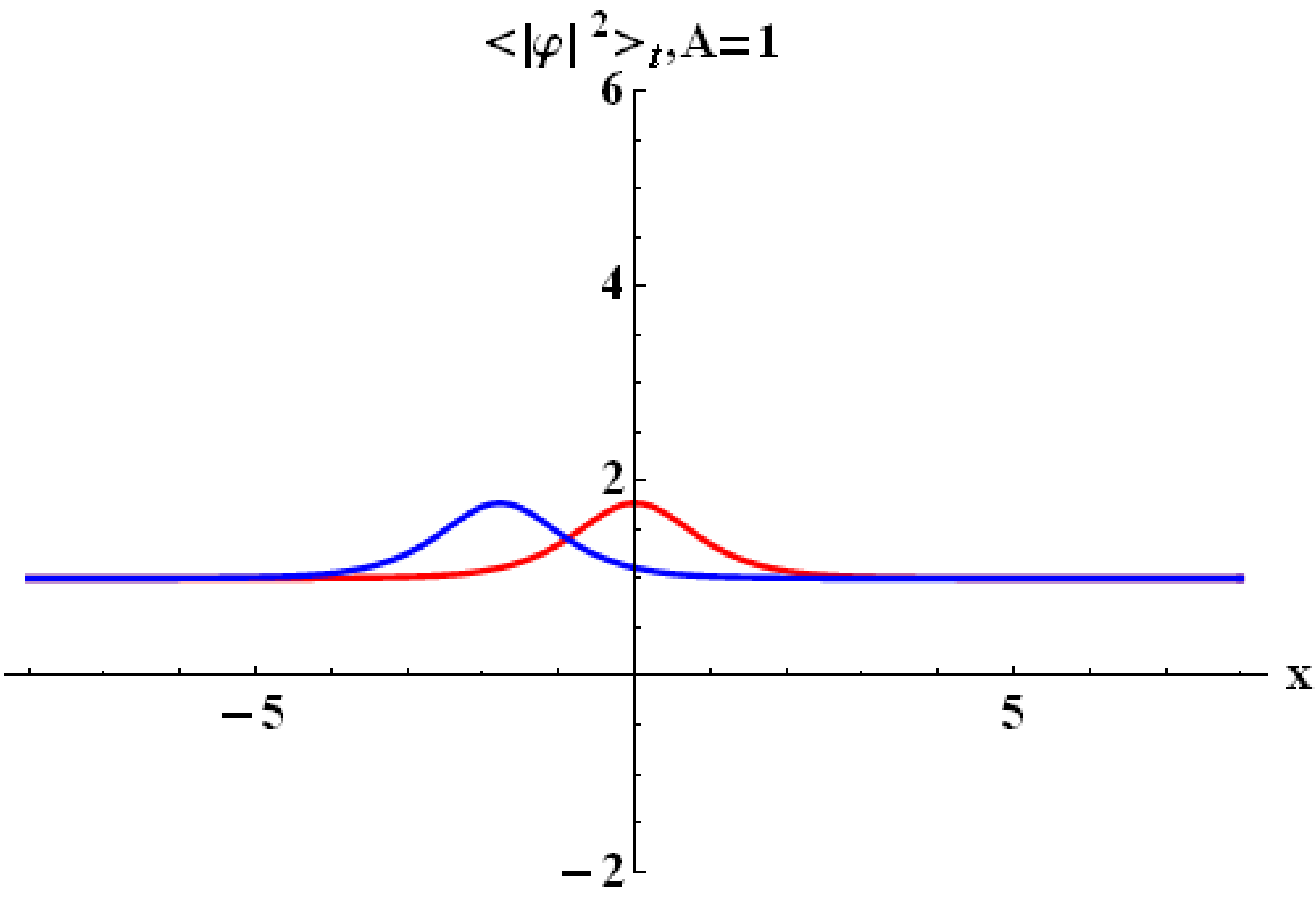
$$+\Big(\cosh(z)\sin(2\alpha)\sinh(2u) + \sinh(2z)\cos(\alpha)\sin(\tau)\Big)^2\Big] d\tau$$

$$= A^2 \left(1 + \frac{4\cosh(2u)}{\left[\cosh^2(2u) - \frac{\cos^2(\alpha)}{\cosh^2(z)}\right]^{3/2}} \frac{\sinh^2(z)\cos^2(\alpha)(\sinh^2(z) + \sin^2(\alpha))}{\cosh^2(z)}\right)$$

$$\left\langle |\varphi|^2 \right\rangle_T = A^2\left(1 + \frac{\sinh^4(z)}{\cosh^2(z)}\right)$$

Kuznetsov-Ma

Time averaging



$$R = 2, \alpha = \frac{5\pi}{16}$$

Two-solitonic solution

$$\varphi = A - 2\frac{N}{\Delta}$$

$$N = \frac{B_1}{\lambda_2 + \lambda_2^*} - \frac{B_2}{\lambda_1^* + \lambda_2} - \frac{B_3}{\lambda_2^* + \lambda_1} + \frac{B_4}{\lambda_1 + \lambda_1^*}$$

$$\Delta = \frac{|q_1|^2 |q_2|^2}{(\lambda_1 + \lambda_1^*)(\lambda_2 + \lambda_2^*)} - \frac{(\vec{q}_1 \vec{q}_2^*)(\vec{q}_1^* \vec{q}_2)}{(\lambda_1^* + \lambda_2)(\lambda_2^* + \lambda_1)}$$

$$B_1 = |q_2|^2 q_{11}^* q_{12} \qquad B_2 = (q_1^* q_2) q_{21}^* q_{12} \qquad B_3 = (q_1 q_2^*) q_{11}^* q_{22} \qquad B_4 = |q_1|^2 q_{21}^* q_{22}$$

We study only regular two-solitonic solution

$$\alpha_2 = -\alpha_1 = -\alpha$$

Hence now

$$C_1 = e^{i\theta_1} \qquad C_2 = e^{i\theta_2}$$

$$\theta^+ = \theta_1 + \theta_2 \qquad \theta^- = \theta_1 - \theta_2$$

$$q_{11} = e^{-\phi_1} + e^{-i\alpha - z_1 + \phi_1} \qquad q_{21} = e^{-\phi_2} + e^{i\alpha - z_2 + \phi_2}$$

$$q_{12} = e^{-i\alpha - z_1 - \phi_1} + e^{\phi_1} \qquad q_{22} = e^{i\alpha - z_2 - \phi_2} + e^{\phi_2}$$

Two-solitonic solution

$$\phi_1 = u_1 + iv_1 \qquad \phi_2 = u_2 + iv_2$$

$$u_1 = æ_1 x - \gamma_1 t \qquad v_1 = k_1 x - \omega_1 t + \frac{1}{2}\theta_1$$

$$u_2 = æ_2 x - \gamma_2 t \qquad v_2 = k_2 x - \omega_2 t + \frac{1}{2}\theta_2$$

$$æ_1 = \frac{A}{2}\left(R_1 - \frac{1}{R_1}\right)\cos(\alpha) = A\sinh(z_1)\cos(\alpha)$$

$$k_1 = \frac{A}{2}\left(R_1 + \frac{1}{R_1}\right)\sin(\alpha) = A\cosh(z_1)\sin(\alpha)$$

$$\gamma_1 = -\frac{A^2}{4}\left(R_1^2 + \frac{1}{R_1^2}\right)\sin(2\alpha) = -\frac{A^2}{2}\cosh(2z_1)\sin(2\alpha)$$

$$\omega_1 = \frac{A^2}{4}\left(R_1^2 - \frac{1}{R_1^2}\right)\cos(2\alpha) = \frac{A^2}{2}\sinh(2z_1)\cos(2\alpha)$$

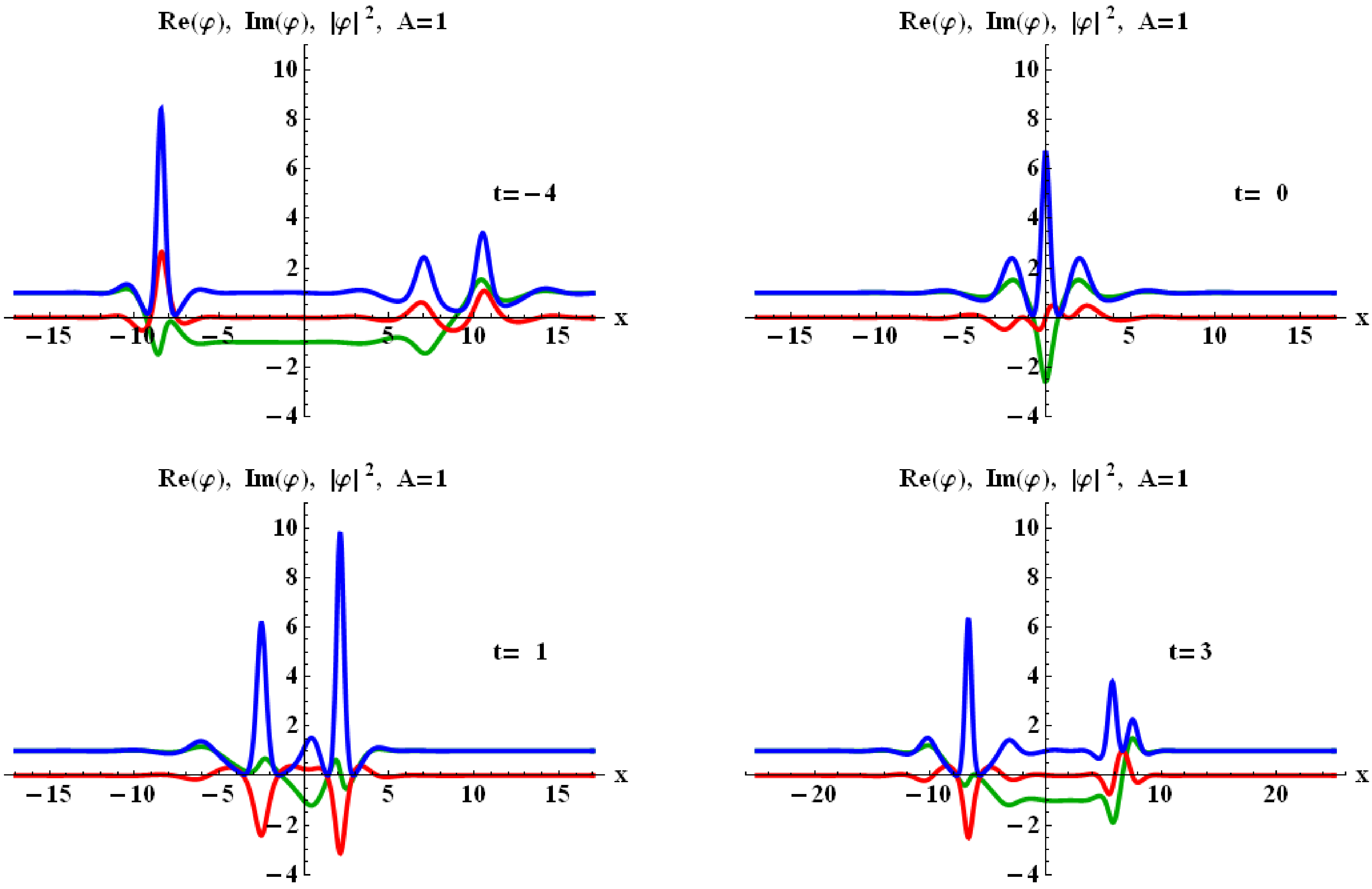
$$æ_2 = \frac{A}{2}\left(R_2 - \frac{1}{R_2}\right)\cos(\alpha) = A\sinh(z_2)\cos(\alpha)$$

$$k_2 = -\frac{A}{2}\left(R_2 + \frac{1}{R_2}\right)\sin(\alpha) = -A\cosh(z_2)\sin(\alpha)$$

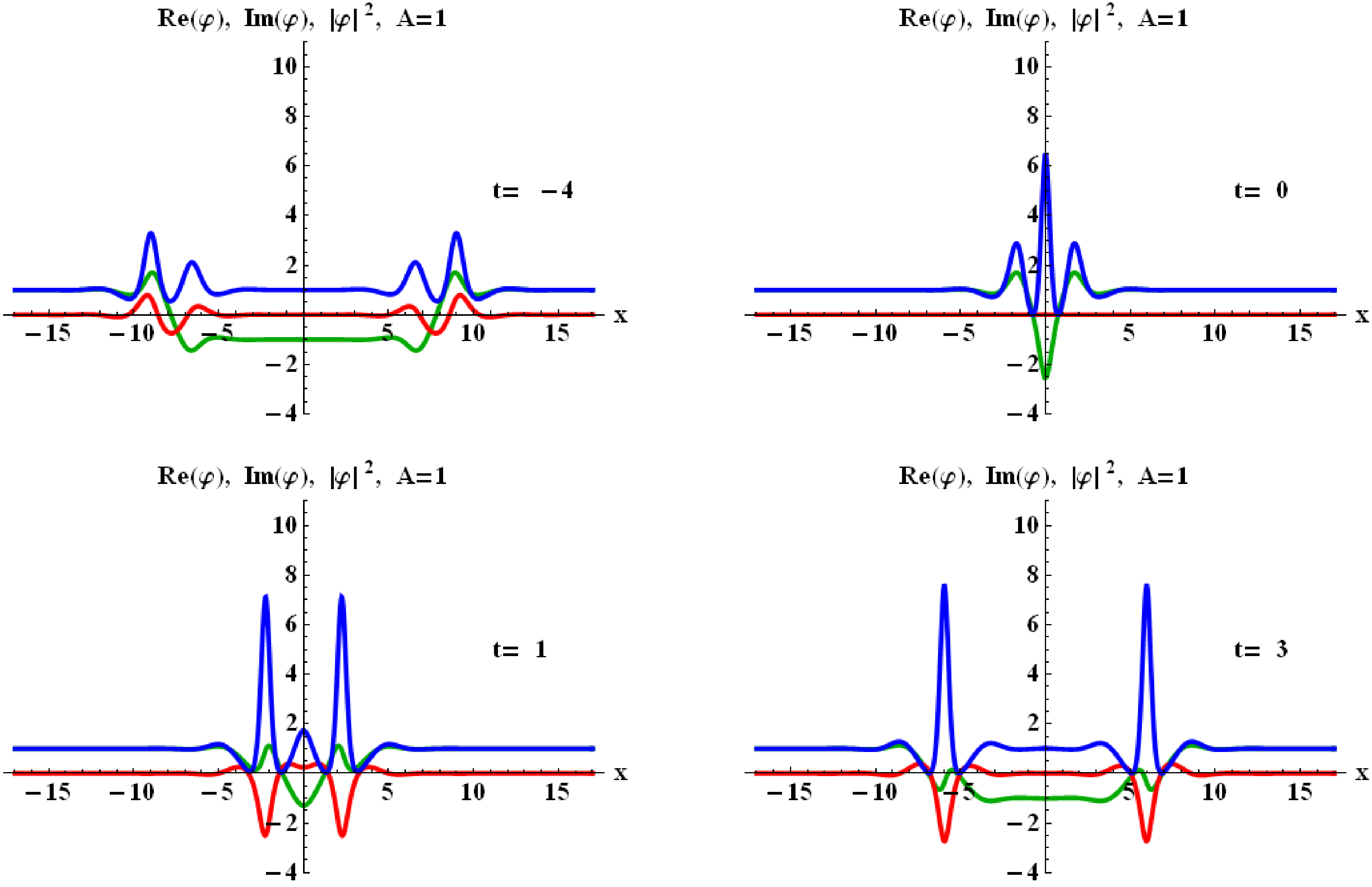
$$\gamma_2 = \frac{A^2}{4}\left(R_2^2 + \frac{1}{R_2^2}\right)\sin(2\alpha) = \frac{A^2}{2}\cosh(2z_2)\sin(2\alpha)$$

$$\omega_2 = \frac{A^2}{4}\left(R_2^2 - \frac{1}{R_2^2}\right)\cos(2\alpha) = \frac{A^2}{2}\sinh(2z_2)\cos(2\alpha)$$

Regular two-solitonic solution



Regular symmetric two-solitonic solution



Regular two-solitonic solution

$$\varphi = A - 2A\frac{M + iK}{H}$$

$$\begin{aligned} H = \; & 4\left[\cos^2(\alpha) + \sinh^2(z)\right]\left[\cosh^2(z)\sin^2(\alpha)\cosh(4æx) - \sinh^2(z)\cos^2(\alpha)\cos(4kx + \theta^-)\right] \\ & + \sinh^2(2z)\cosh(4\gamma t) - \sin^2(2\alpha)\cos(4\omega t - \theta^+) \\ & + 2\sinh(2z)\sin(2\alpha)\sinh(z)\sin(\alpha) \times \\ & \left[\cosh(2æx - 2\gamma t)\cos(2kx + 2\omega t - \theta_2) + \cosh(2æx + 2\gamma t)\cos(2kx - 2\omega t + \theta_1)\right] \\ & + 2\sinh(2z)\sin(2\alpha)\cosh(z)\cos(\alpha) \times \\ & \left[\sinh(2æx - 2\gamma t)\sin(2kx + 2\omega t - \theta_2) + \sinh(2æx + 2\gamma t)\sin(2kx - 2\omega t + \theta_1)\right] \end{aligned}$$

Regular two-solitonic solution

$$M = \sinh(2z)\sin(2\alpha) \times$$

$$\Big(\sinh(2z)\sin(2\alpha)\left[\cosh(4\gamma t) + \cos(4\omega t - \theta^{+})\right]$$

$$+2\sinh(z)\sin(\alpha)\left[\cos^2(\alpha) + \cosh^2(z)\right] \times$$

$$\left[\cosh(2æx - 2\gamma t)\cos(2kx + 2\omega t - \theta_2) + \cosh(2æx + 2\gamma t)\cos(2kx - 2\omega t + \theta_1)\right]$$

$$+2\cosh(z)\cos(\alpha)\left[\sin^2(\alpha) - \sinh^2(z)\right] \times$$

$$\left[\sinh(2æx - 2\gamma t)\sin(2kx + 2\omega t - \theta_2) + \sinh(2æx + 2\gamma t)\sin(2kx - 2\omega t + \theta_1)\right]\Big)$$

$$K = \sinh(2z)\sin(2\alpha) \times$$

$$\Big(\sinh(2z)\sin(2\alpha)\sinh(4\gamma t) - \sin(2\alpha)\sin(4\omega t - \theta^{+})$$

$$-2\cosh(z)\sin(\alpha)\left[\cos^2(\alpha) + \sinh^2(z)\right] \times$$

$$\left[\cosh(2æx - 2\gamma t)\sin(2kx + 2\omega t - \theta_2) - \cosh(2æx + 2\gamma t)\sin(2kx - 2\omega t + \theta_1)\right]$$

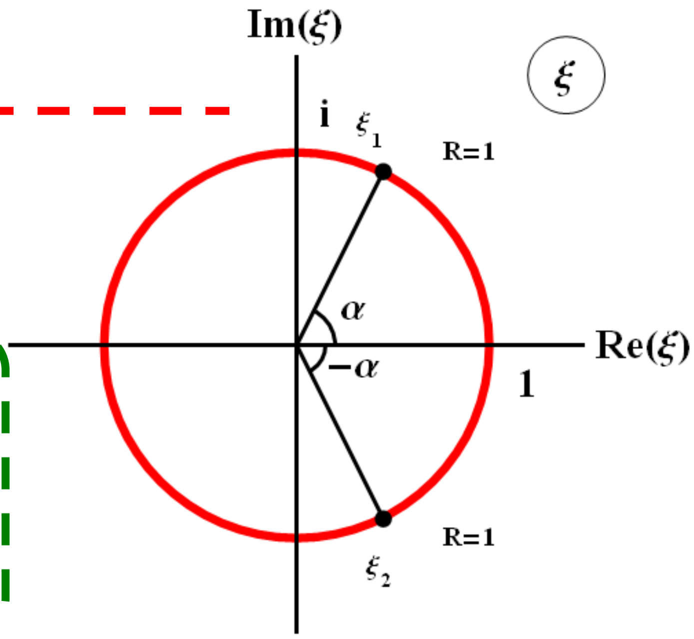
$$-2\sinh(z)\cos(\alpha)\left[\cos^2(\alpha) + \sinh^2(z)\right] \times$$

$$\left[\sinh(2æx - 2\gamma t)\cos(2kx + 2\omega t - \theta_2) - \sinh(2æx + 2\gamma t)\cos(2kx - 2\omega t + \theta_1)\right]$$

Annulation of solitons

Let us consider the limiting case

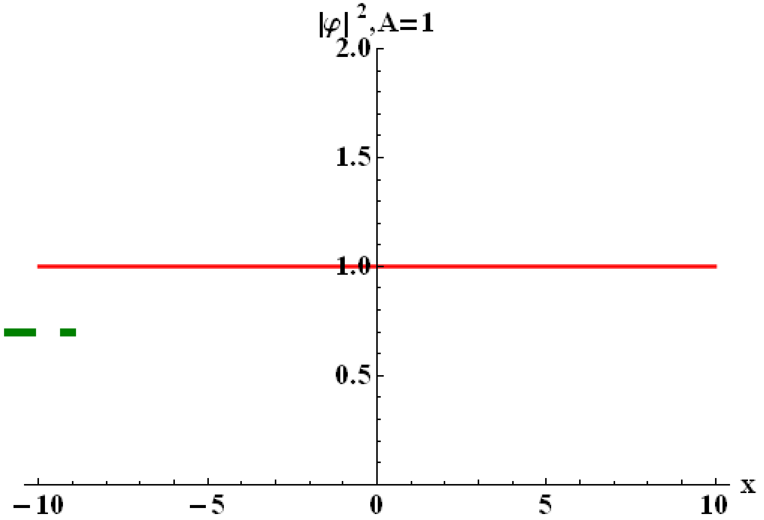
$$R_1 = R_2 = 1$$



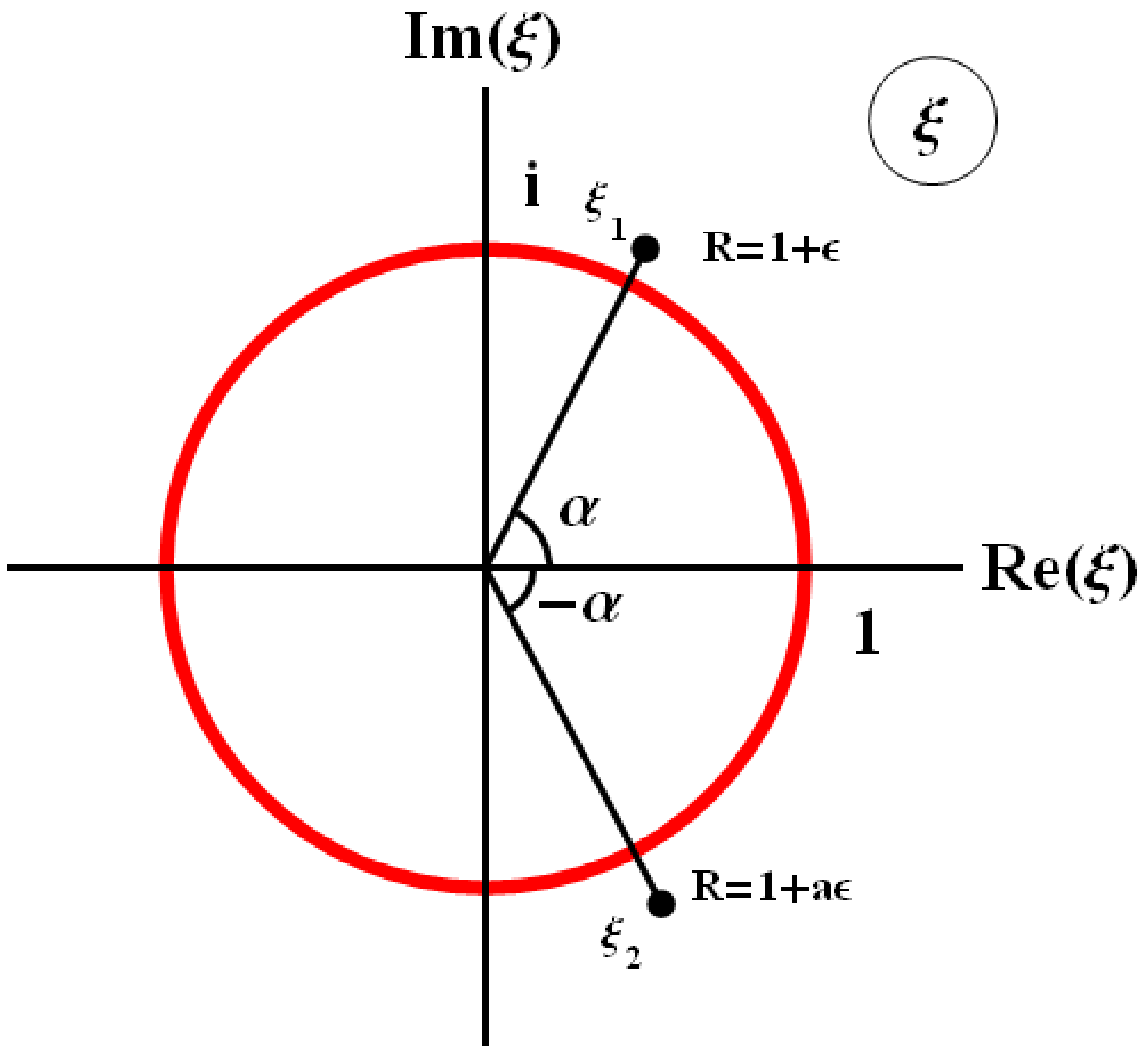
$$\varphi = A - 2\frac{N}{\Delta}$$

$$N = 0 \qquad \Delta = \frac{\sin^2(\alpha)}{A^2\cos^2(\alpha)}\sin^2(\theta^+)$$

If $C_1 C_2 \neq 1$, $\Delta \neq 0$ and $\varphi = A$



Small perturbation of condensate



Small perturbation of condensate

If $R_1 \neq 1, R_2 \neq 1$ interference of solitons is not complete, and the dressing is not trivial. It is given by expression:

$$\delta\varphi = -2\frac{\delta N}{\tilde{\Delta}}$$

now:

$$\lambda_1 \approx A\cos(\alpha) + i\varepsilon A\sin(\alpha) \qquad \lambda_2 \approx A\cos(\alpha) - i\varepsilon a A\sin(\alpha)$$

$$\delta N = \frac{i(a+1)\varepsilon\sin(\alpha)}{4A\cos^2(\alpha)}(B_3 - B_2) \qquad \tilde{\Delta} = \frac{\left|q_{11}q_{22} - q_{12}q_{21}\right|^2}{4A^2\cos^2(\alpha)}$$

Small perturbation of condensate

$$\varphi = A - \varepsilon Ai(1+a)\frac{\begin{pmatrix}\cosh(\xi\varepsilon a x + i\alpha)\sin(\Psi x - \theta_1) \\ -\cosh(\xi\varepsilon x - i\alpha)\sin(\Psi x + \theta_2) - \sin(S)\end{pmatrix}}{\cosh((1+a)\xi\varepsilon x) - \cos(S)}$$

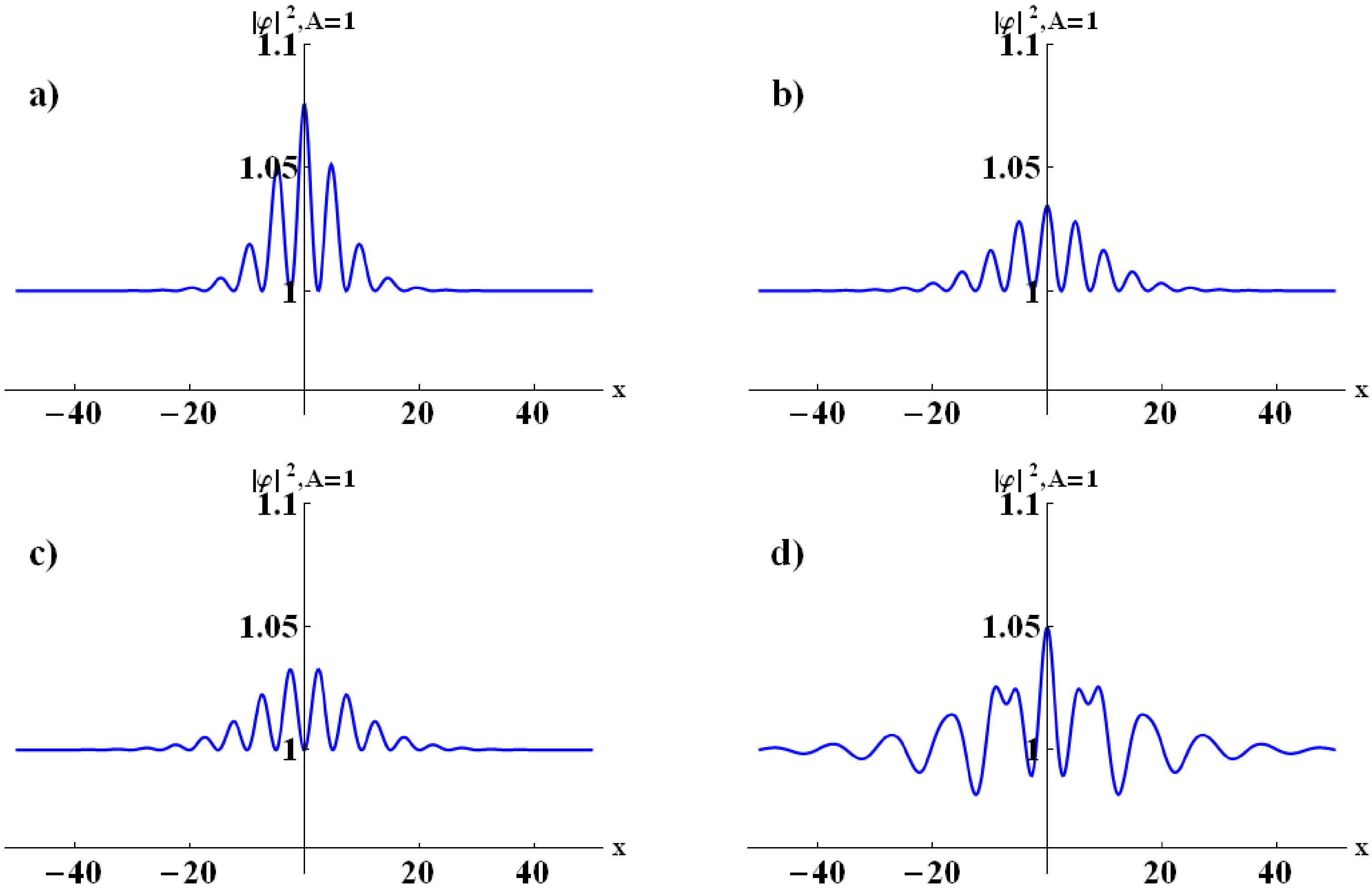
Here:

$$\xi = -2A\cos(\alpha) \quad \Psi = 2A\sin(\alpha)$$

One can simplify φ at $\theta_1 = \theta_2 = \dfrac{\pi}{2}, \quad a = 1:$

$$\varphi = A + 4\varepsilon Ai\frac{\cosh(\xi\varepsilon x)\cos(\Psi x)\cos(\alpha)}{\cosh(2\xi\varepsilon x) + 1}$$

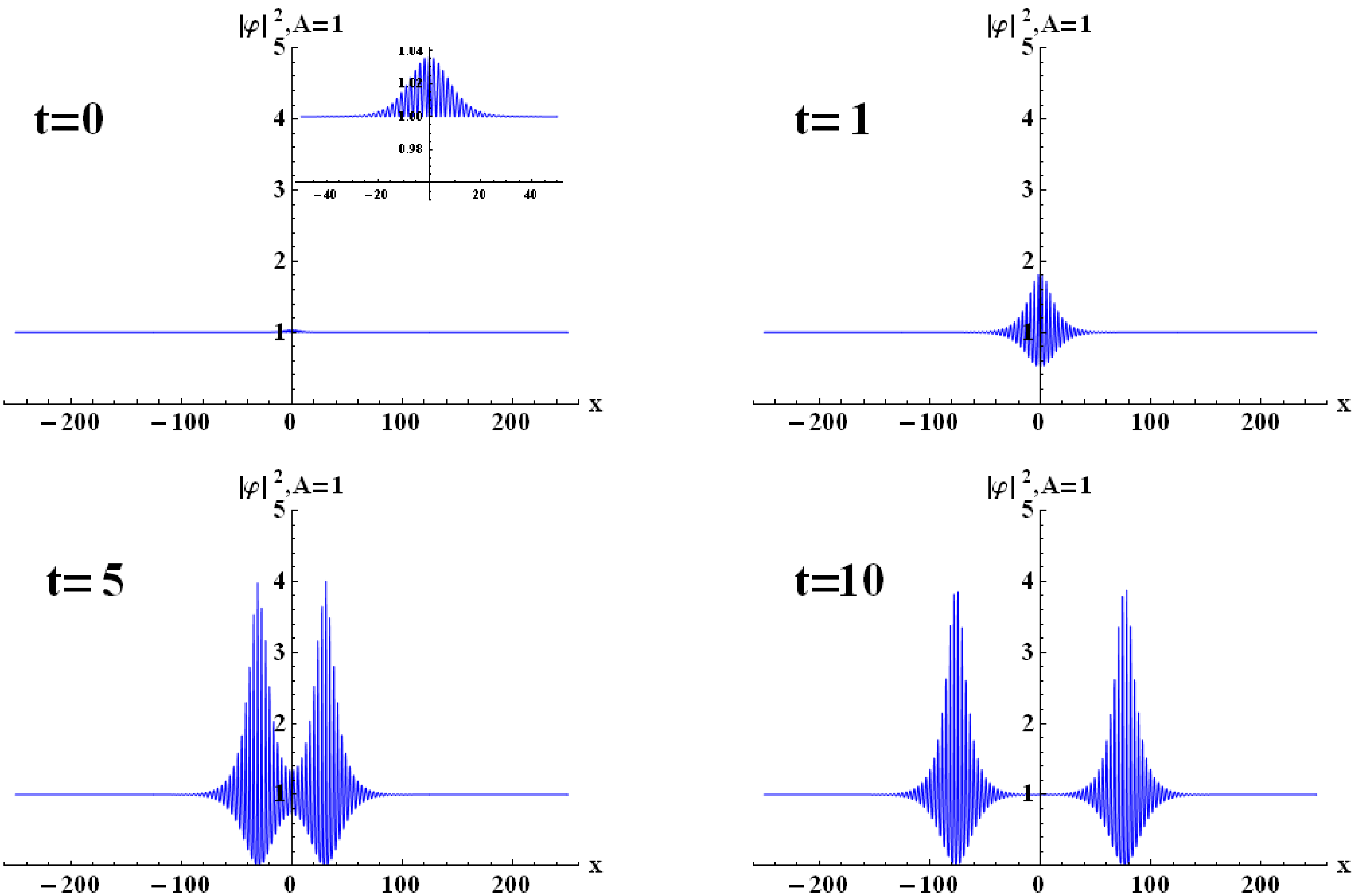
Small perturbation of condensate



$$a) \quad R = 1.075, \ \alpha = \frac{\pi}{10}, \ \theta^{+} = \pi, \quad \theta^{-} = 0$$

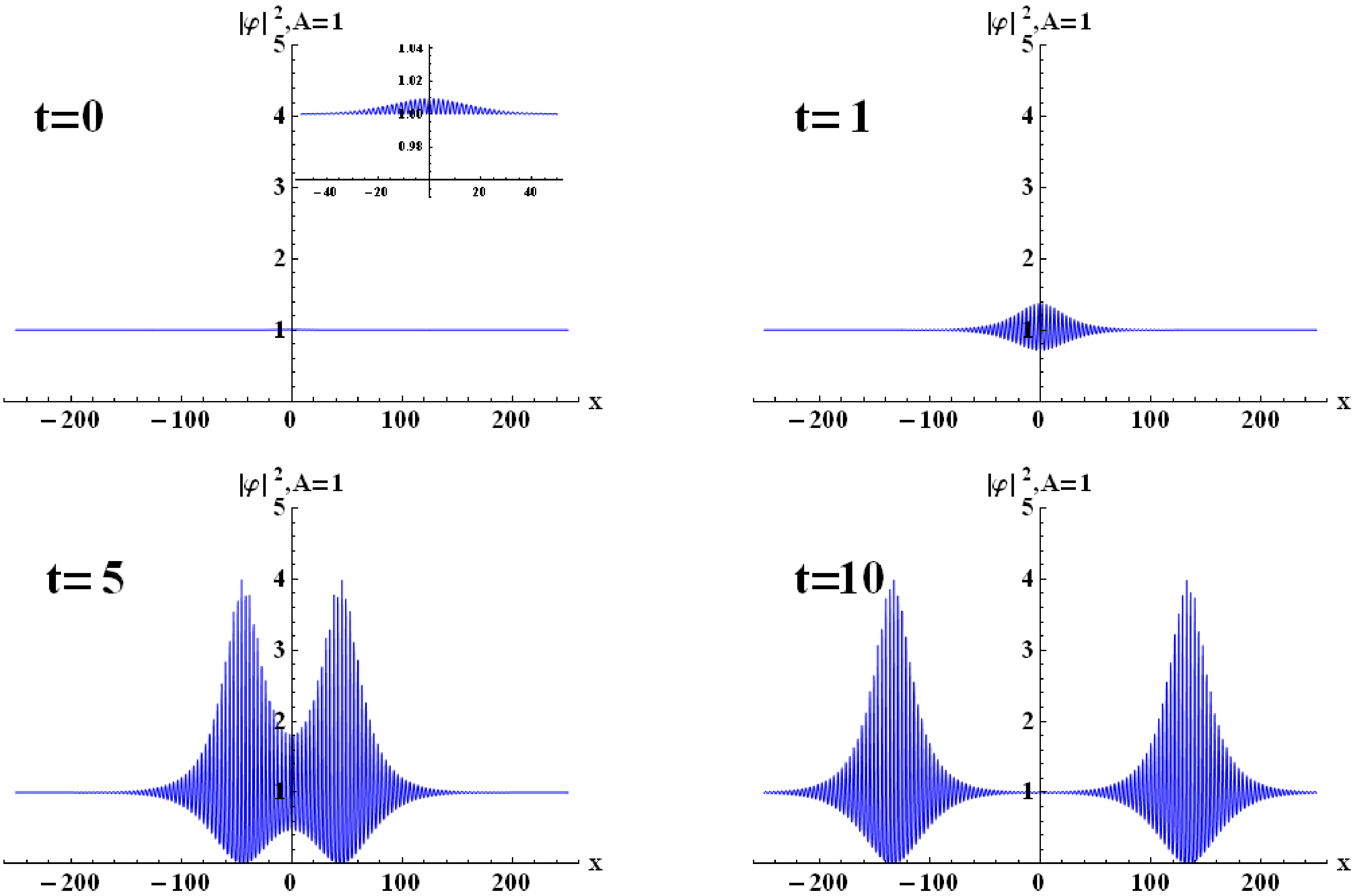
$$b,c,d) \quad R = 1.05, \ \alpha = \frac{\pi}{10} \quad b)\, \theta^{+} = \pi, \theta^{-} = 0 \ \ c)\, \theta^{+} = \pi, \theta^{-} = \pi \ \ c)\, \theta^{+} = \frac{4}{5}\pi, \theta^{-} = 0$$

Small perturbation of condensate



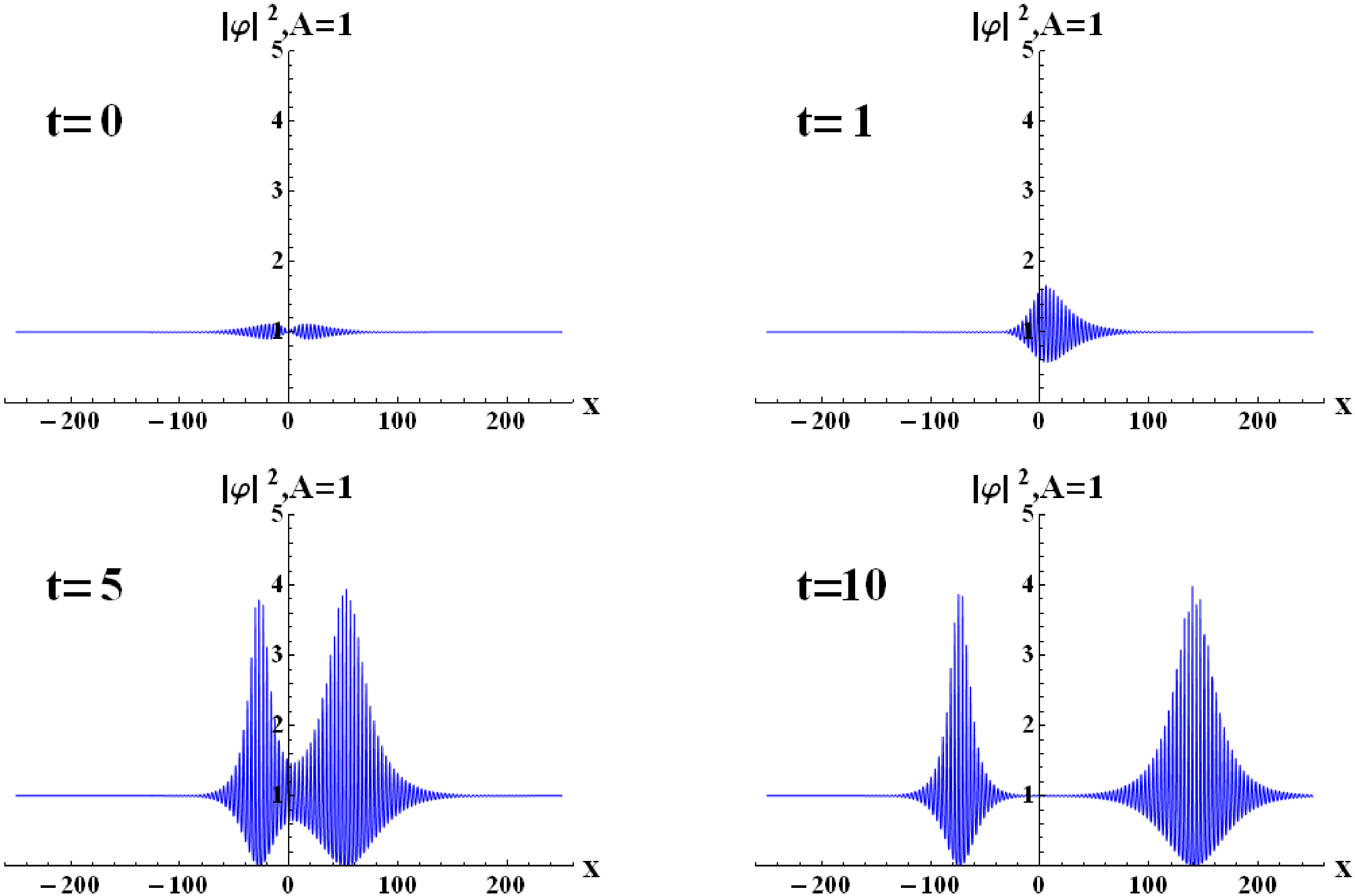
$$R_1 = R_2 = 1.1, \quad \alpha_1 = \frac{\pi}{3}, \ \alpha_2 = -\frac{\pi}{3}, \quad C_1 = C_2 = \frac{\pi}{2}$$

Small perturbation of condensate



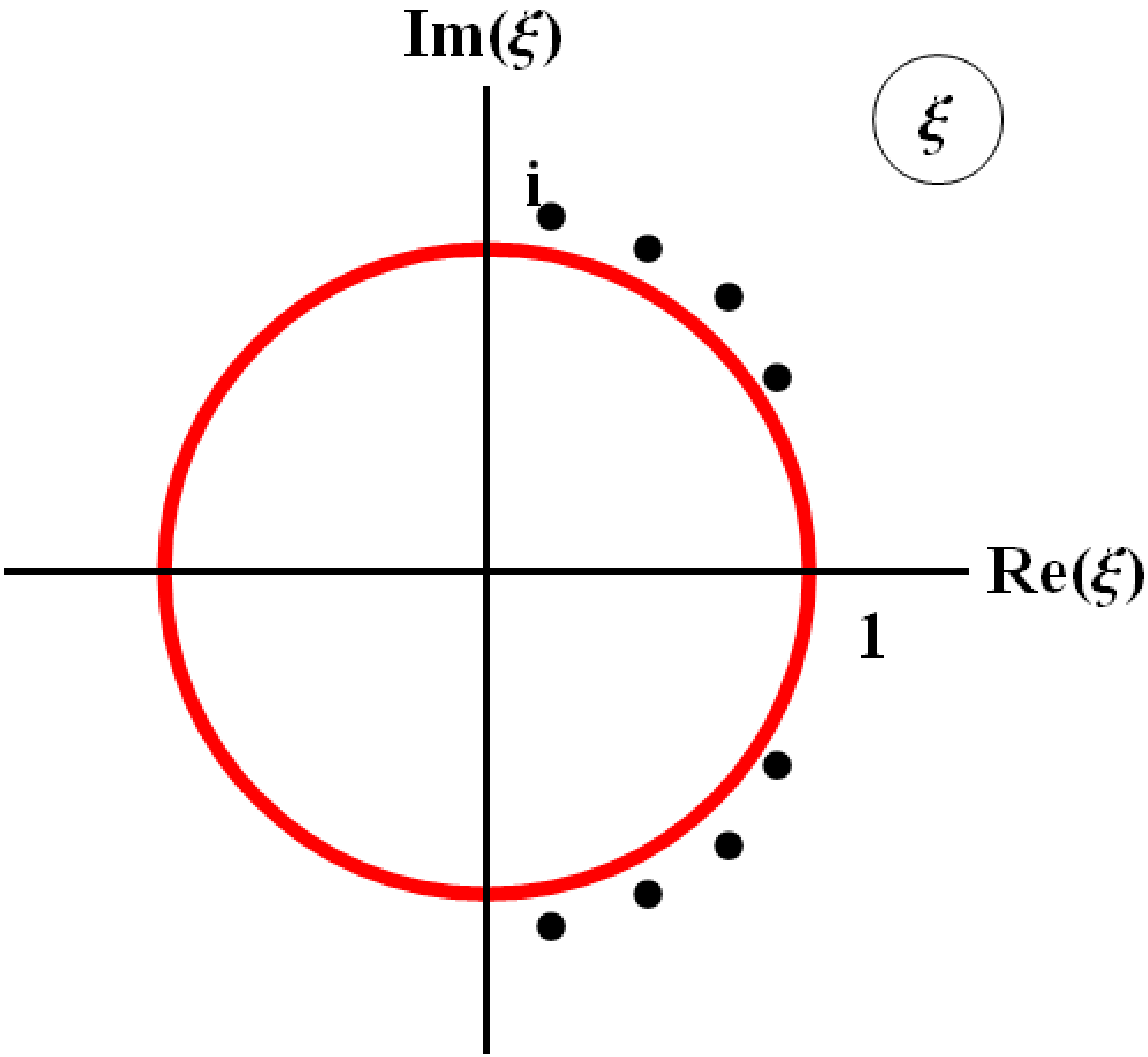
$$R_1 = R_2 = 1.05, \quad \alpha_1 = \frac{\pi}{3}, \ \alpha_2 = -\frac{\pi}{3}, \quad C_1 = C_2 = \frac{\pi}{2}$$

Small perturbation of condensate

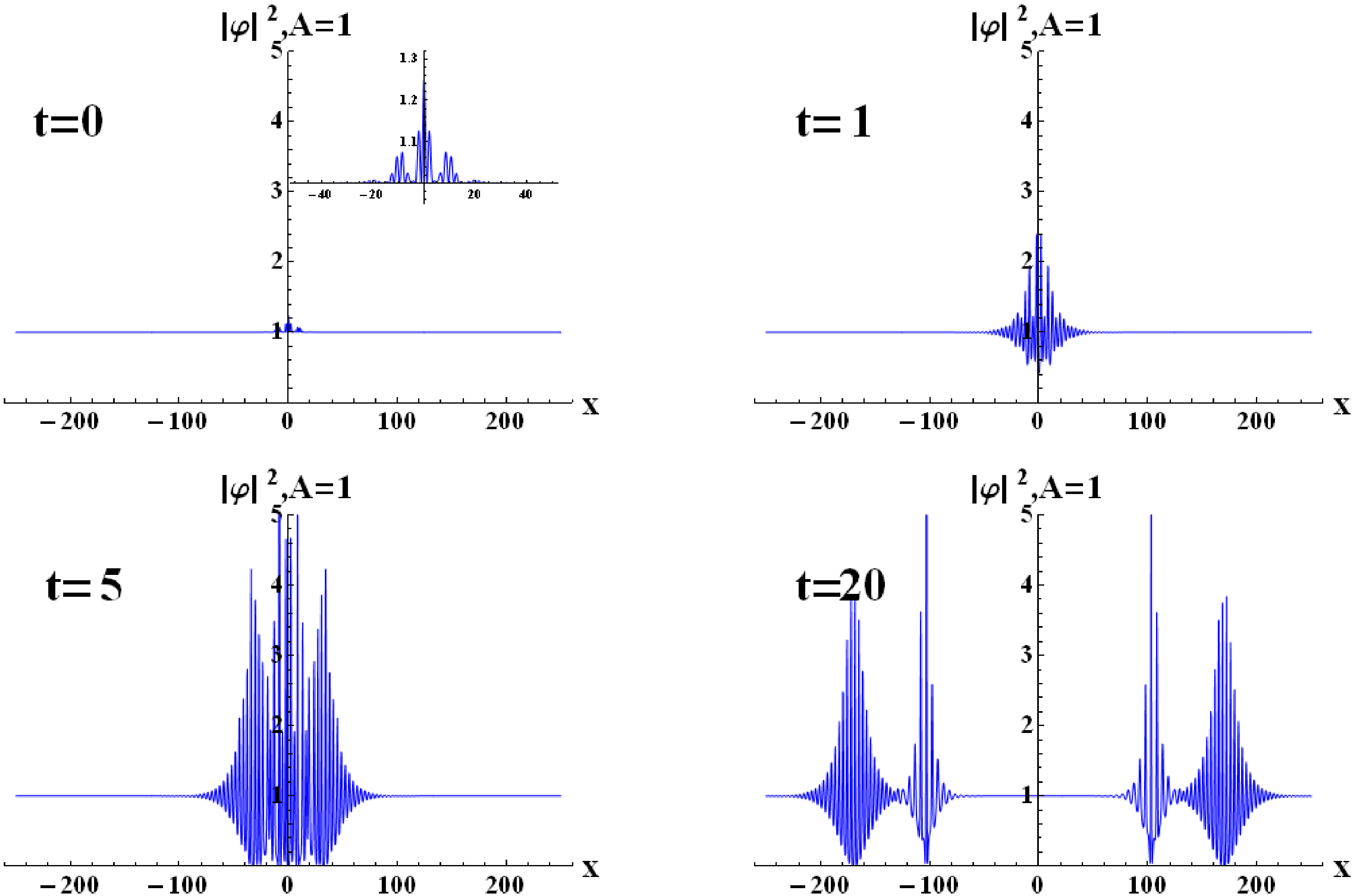


$$R_1 = 1.1,\ R_2 = 1.05, \quad \alpha_1 = \frac{\pi}{3},\ \alpha_2 = -\frac{\pi}{3}, \quad C_1 = C_2 = \frac{\pi}{2}$$

Small perturbation of condensate (2N-solitonic case)



Small perturbation of condensate (2N-solitonic case)



$$R_1 = R_2 = R_3 = R_4 = 1.1, \quad \alpha_1 = \frac{\pi}{3},\ \alpha_2 = -\frac{\pi}{3}, \alpha_1 = \frac{\pi}{5},\ \alpha_2 = -\frac{\pi}{5}, \qquad C_1 = C_2 = C_3 = C_4 = \frac{\pi}{2}$$

Welcome to snowy Akademgorodok!

Solitonc atom